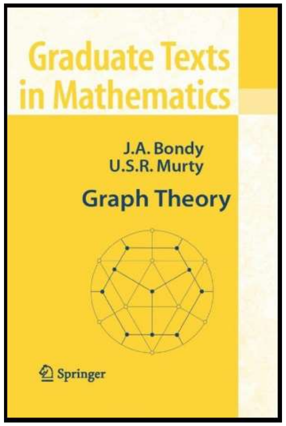
Graph Theory

Chapter 14. Vertex Colourings

14.1. Chromatic Number—Proofs of Theorems



Lemma 14.1.A

Lemma 14.1.A. The number of colours needed in the Greedy Colouring Heuristic (Heuristic 14.3) is at most $\Delta + 1$. Therefore, for any graph G, $\chi(G) \leq \Delta + 1$.

Proof. Let G be a graph. At Step 2 of the heuristic, when vertex v is about to be coloured, the number of its neighbors already coloured is at most $d(v) \leq \Delta$. So one of the colours $1, 2, \ldots, \Delta + 1$ is available to be assigned to v. Since v is an arbitrary vertex of G, then colours $1, 2, \ldots, \Delta + 1$ are sufficient for the heuristic. So any graph G is $\Delta + 1$ colourable and $\chi(G) \leq \Delta + 1$, as claimed. □

Theorem 14.4. Books' Theorem

Theorem 14.4. Brooks' Theorem.
If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof. First, we consider the case that G is not regular. Let x be a vertex of (minimum) degree δ and let T be a DFS-tree of G rooted at x. We colour the vertices with colours $1, 2, \ldots, \Delta$ using the Greedy Colouring Heuristic (Heuristic 14.3). To apply it, we select at each step a leaf of the subtree of T induced by the vertices not yet coloured. That is, we use the spanning DFS-tree T to imply a linear ordering of the vertices of G based on this sequential idea of using leaves in subtrees of T. Notice that the root will be the last vertex in the ordering.

When a vertex $v \neq x$ is about to be coloured by the heuristic, it is adjacent in T to at least one uncoloured vertex and so is adjacent in G to at most $d(v) - 1$ coloured vertices (and $d(v) - 1 \leq \Delta - 1$). So vertex $v \neq x$ can be assigned one of the colours $1, 2, \ldots, \Delta$.

Theorem 14.4 (continued 1)

Proof (continued). Hence $G - x$ can be properly coloured with Δ colours. We chose root x to satisfy $d(x) = \delta \leq \Delta - 1$ (notice that $\delta \neq \Delta$ since G is not regular; this is where we use the fact that G is not regular). So x can also be assigned one of the colours $1, 2, \ldots, \Delta$. So the Greedy Colouring Heuristic applied in this way (with the linear ordering given by the DFS-tree in fact, any spanning tree rooted at a vertex of degree δ could be used here) produces a Δ-colouring of G. So if G is not regular, then the claim holds.

Second we consider the case that G is regular. If G has a cut vertex x then $G = G_1 \cup G_2$ where G_1 and G_2 are connected and $G_1 \cap G_2 = \{x\}$. Because the degree of x in G_1 and in G_2 is less that $\Delta(G)$ (since G is regular then all vertices of G are of degree Δ), so neither G_1 not G_2 is regular. So by the "not regular" case above, $\chi(G_i) \leq \Delta(G_i) = \Delta(G)$ for $i \in \{a, 2\}$. Then by Exercise 14.1.2 we have $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} \leq \Delta(G)$. So the result holds for G regular with a cut vertex.

Theorem 14.4 (continued 2)

Proof (continued). Hence, without loss of generality we can assume that G is regular and does not have a cut vertex (that is, G is regular and 2-connected).

If every DFS-tree of G is a Hamilton path rooted at one of its ends, then by Exercise 6.1.11 (as mentioned above in the notes) G is a cycle, a complete graph, or a complete bipartite graph $K_{n,n}$. By hypothesis, G is neither an odd cycle nor a complete graph. Since $\chi(K_{n,n}) = 2 \leq \Delta(G)$ then the result holds for such graphs. (We use the fact that T is a DFS-tree for the first time here.)

Finally, suppose G is regular, 2-connected, that T is a DFS-tree of G, but that T is not a path. Let x be a vertex of T with at least two children, y and z (such x exists since T is not a path; for the genealogical terminology, which is based on the rooted spanning tree, see Section 6.1. Tree Search). Since G is 2-connected, then both $g - y$ and $G - z$ are connected.

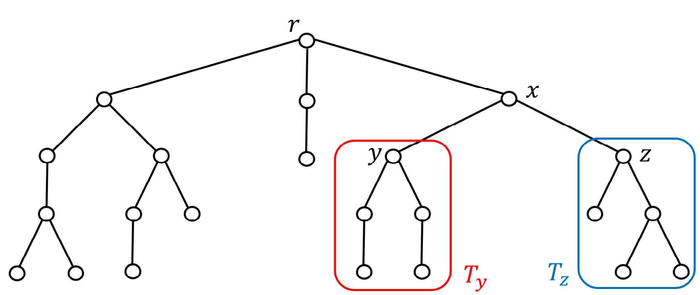
Theorem 14.4 (continued 3)

Theorem 14.4. Brooks' Theorem.
If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof (continued). Thus y and z are either leaves of T or have proper descendants in T. If y (respectively z) is a leaf of T then $G - y$ (respectively $G - z$) is connected. In the event that y or z have proper descendants in T, we partition the vertices of G into three sets: those that are descendants of y, those that are descendants of z, and those that are neither descendants of y nor descendants of z. These three sets induce three subtrees of T: the subtree T_y of T rooted at y induced by the descendants of y, the subtree T_z of T rooted at z induced by the descendants of z, and the subtree T_r of T induced by the other vertices. Notice that T_r results from T by "pruning" T by removing edge xy and T_x, and removing edge xz and T_z. See the figure on the next slide.

Theorem 14.4 (continued 4)

Proof (continued).



In Exercise 14.1.A it is to be shown, using T_r, T_y, and T_z, that $G' = G - \{y, z\}$ is connected. Since G is regular and to create G, we removed two vertices from G, then the most the degree of a vertex can decrease from G to G' is 2, and x satisfies $d_G(x) - 2 = d_{G'}(x)$ so that x is a vertex of (minimum) degree δ in G'. Also, G' is not regular. Let T' by a DFS-tree rooted at x in G'.

Theorem 14.4 (continued 5)

Theorem 14.4. Brooks' Theorem.
If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof (continued). Notice that y and z are not related (recall that two vertices are "related" if one is an ancestor of the other) in T. By Theorem 6.6, for T a DFS-tree of G every edge of G joins vertices which are related, so y and z are not adjacent in G. By the first part of the proof, the Greedy Colouring Heuristic lets us colour the vertices of T', ending with the root x, giving a Δ-colouring of G'. Since y and z are not adjacent adjacent in G, then $d_G(y) \leq \Delta - 1$ and $d_G(z) \leq \Delta - 1$, so we can assign a colour $1, 2, \ldots, \Delta$ to y and such a colour to z, yielding a Δ-colouring of G. So the case for G a regular graph, the claim holds. Therefore, the claim holds for all connected graphs that are neither an odd cycle nor a complete graph. □

Theorem 14.5. The Gallai-Roy Theorem

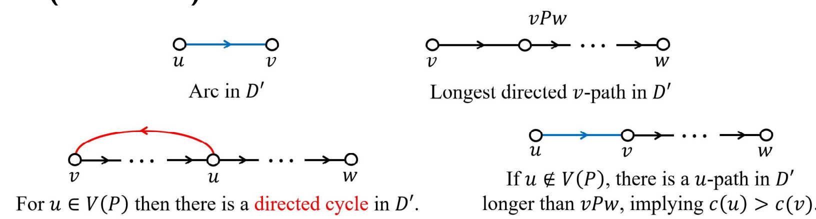
Theorem 14.5. The Gallai-Roy Theorem.
Every digraph D contains a directed path with $\chi(D)$ vertices.

Proof. Let k be the number of vertices in a longest directed path of D. Consider a maximal acyclic subgraph D' of D (i.e., D' does not contain a directed cycle and is a subgraph of D with the most arcs which satisfies this property). Because D' is a subgraph of D, each directed path in D' has at most k vertices. We k-colour D by assigning to vertex v the colour $c(v)$, where $c(v) \in \{1, 2, \ldots, k\}$ is the number of vertices of a longest directed path in D' starting at v. We claim this vertex colouring is proper.

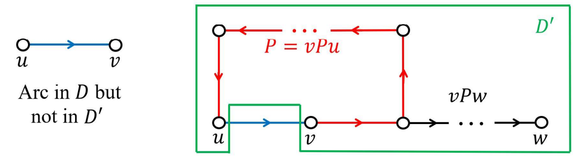
Let (u, v) be an arbitrary arc of D. (1) If (u, v) is an arc of D' then let vPw be a longest directed v-path in D'. If $u \in V(P)$ then there would be a directed cycle in D' (see the figure below), hence $u \notin V(P)$. Thus $uvPw$ is a directed u-path in D' and we have $c(u) > c(v)$.

Theorem 14.5 (continued 1)

Proof (continued).



(2) If (u, v) is not an arc of D', then $D' + (u, v)$ contains a directed cycle (because D' is maximally acyclic), so D' contains a directed (v, u)-path P.

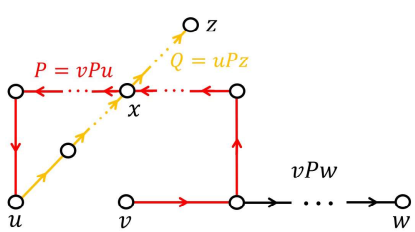


Since D' is maximally acyclic then $D' + (u, v)'$ contains a directed cycle.

Theorem 14.5 (continued 2)

Theorem 14.5. The Gallai-Roy Theorem.
Every digraph D contains a directed path with $\chi(D)$ vertices.

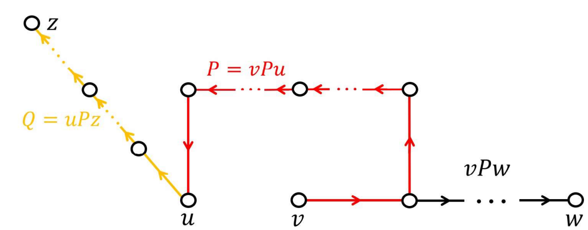
Proof (continued). Let Q be a longest directed u-path in D'. Because D' is acyclic, $V(P) \cap V(Q) = \{u\}$ (for if there were a second vertex x in $V(P) \cap V(Q)$ then there would be a directed cycle in D' from u, along Q to w, then along P back to u; see the figure below).



If $u \neq x \in V(P) \cap V(Q)$ then there is a directed cycle in D'.

Theorem 14.5 (continued 3)

Theorem 14.5. The Gallai-Roy Theorem.
Every digraph D contains a directed path with $\chi(D)$ vertices.
Proof (continued).



Directed v-path $vPuQz = PQ$ in D' is longer than the longest u-path, Q, in D'.

Thus PQ is a directed v path in D' longer than Q (see the figure above), so that $c(v) > c(u)$. In both cases, $c(u) \neq c(v)$. Since (u, v) is an arbitrary arc of D, then the colouring is proper. Since $k \geq \chi(D)$, then D has a directed path of length (at least) $\chi(D)$, as claimed. □