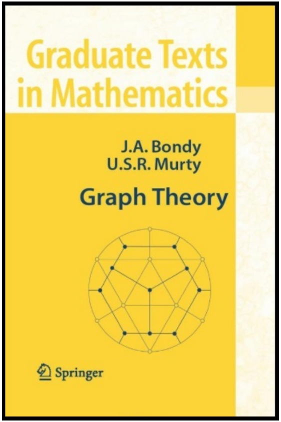
Graph Theory

Chapter 14. Vertex Colourings

14.1. Chromatic Number—Proofs of Theorems



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Lemma 14.1.A

Lemma 14.1.A. The number of colours needed in the Greedy Colouring Heuristic (Heuristic 14.3) is at most $\Delta + 1$. Therefore, for any graph G, $\chi(G) \leq \Delta + 1$.

Proof. Let G be a graph. At Step 2 of the heuristic, when vertex v is about to be coloured, the number of its neighbors already coloured is at most $d(v) \leq \Delta$. So one of the colours $1, 2, \ldots, \Delta + 1$ is available to be assigned to v. Since v is an arbitrary vertex of G, then colours $1, 2, \ldots, \Delta + 1$ are sufficient for the heuristic. So any graph G is $\Delta + 1$ colourable and $\chi(G) \leq \Delta + 1$, as claimed. □

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Theorem 14.4. Books' Theorem

Theorem 14.4. Brooks' Theorem.
If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof. First, we consider the case that G is not regular. Let x be a vertex of (minimum) degree δ and let T be a DFS-tree of G rooted at x. We colour the vertices with colours $1, 2, \ldots, \Delta$ using the Greedy Colouring Heuristic (Heuristic 14.3). To apply it, we select at each step a leaf of the subtree of T induced by the vertices not yet coloured. That is, we use the spanning DFS-tree T to imply a linear ordering of the vertices of G based on this sequential idea of using leaves in subtrees of T. Notice that the root will be the last vertex in the ordering.

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When a vertex $v \neq x$ is about to be coloured by the heuristic, it is adjacent in T to at least one uncoloured vertex and so is adjacent in G to at most $d(v) - 1$ coloured vertices (and $d(v) - 1 \leq \Delta - 1$). So vertex $v \neq x$ can be assigned one of the colours $1, 2, \ldots, \Delta$.

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Theorem 14.4 (continued 1)

Proof (continued). Hence $G - x$ can be properly coloured with Δ colours. We chose root x to satisfy $d(x) = \delta \leq \Delta - 1$ (notice that $\delta \neq \Delta$ since G is not regular; this is where we use the fact that G is not regular). So x can also be assigned one of the colours $1, 2, \ldots, \Delta$. So the Greedy Colouring Heuristic applied in this way (with the linear ordering given by the DFS-tree in fact, any spanning tree rooted at a vertex of degree δ could be used here) produces a Δ-colouring of G. So if G is not regular, then the claim holds.

Second we consider the case that G is regular. If G has a cut vertex x then $G = G_1 \cup G_2$ where G_1 and G_2 are connected and $G_1 \cap G_2 = \{x\}$. Because the degree of x in G_1 and in G_2 is less that $\Delta(G)$ (since G is regular then all vertices of G are of degree Δ), so neither G_1 not G_2 is regular. So by the "not regular" case above, $\chi(G_i) \leq \Delta(G_i) = \Delta(G)$ for $i \in \{a, 2\}$. Then by Exercise 14.1.2 we have $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} \leq \Delta(G)$. So the result holds for G regular with a cut vertex.

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Proof (continued). Hence, without loss of generality we can assume that G is regular and does not have a cut vertex (that is, G is regular and 2-connected).

If every DFS-tree of G is a Hamilton path rooted at one of its ends, then by Exercise 6.1.11 (as mentioned above in the notes) G is a cycle, a complete graph, or a complete bipartite graph $K_{n,n}$. By hypothesis, G is neither an odd cycle nor a complete graph. Since $\chi(K_{n,n}) = 2 \leq \Delta(G)$ then the result holds for such graphs. (We use the fact that T is a DFS-tree for the first time here.)

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Finally, suppose G is regular, 2-connected, that T is a DFS-tree of G, but that T is not a path. Let x be a vertex of T with at least two children, y and z (such x exists since T is not a path; for the genealogical terminology, which is based on the rooted spanning tree, see Section 6.1. Tree Search). Since G is 2-connected, then both $g - y$ and $G - z$ are connected.

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Theorem 14.4. Brooks' Theorem.
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Proof (continued). Thus y and z are either leaves of T or have proper descendants in T. If y (respectively z) is a leaf of T then $G - y$ (respectively $G - z$) is connected. In the event that y or z have proper descendants in T, we partition the vertices of G into three sets: those that are descendants of y, those that are descendants of z, and those that are neither descendants of y nor descendants of z. These three sets induce three subtrees of T: the subtree T_y of T rooted at y induced by the descendants of y, the subtree T_z of T rooted at z induced by the descendants of z, and the subtree T_r of T induced by the other vertices. Notice that T_r results from T by "pruning" T by removing edge xy and T_x, and removing edge xz and T_z. See the figure on the next slide.

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Theorem 14.4 (continued 4)

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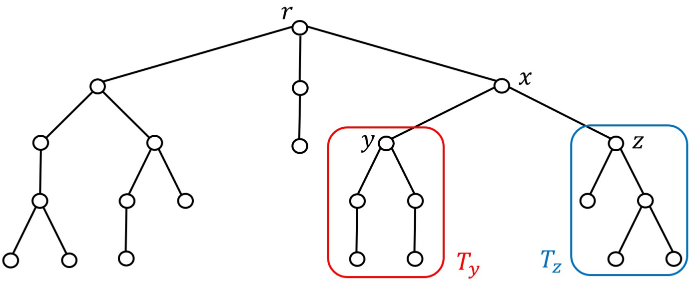

In Exercise 14.1.A it is to be shown, using T_r, T_y, and T_z, that $G' = G - \{y, z\}$ is connected. Since G is regular and to create G, we removed two vertices from G, then the most the degree of a vertex can decrease from G to G' is 2, and x satisfies $d_G(x) - 2 = d_{G'}(x)$ so that x is a vertex of (minimum) degree δ in G'. Also, G' is not regular. Let T' by a DFS-tree rooted at x in G'.

Theorem 14.4 (continued 5)

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If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Proof (continued). Notice that y and z are not related (recall that two vertices are "related" if one is an ancestor of the other) in T. By Theorem 6.6, for T a DFS-tree of G every edge of G joins vertices which are related, so y and z are not adjacent in G. By the first part of the proof, the Greedy Colouring Heuristic lets us colour the vertices of T', ending with the root x, giving a Δ-colouring of G'. Since y and z are not adjacent adjacent in G, then $d_G(y) \leq \Delta - 1$ and $d_G(z) \leq \Delta - 1$, so we can assign a colour $1, 2, \ldots, \Delta$ to y and such a colour to z, yielding a Δ-colouring of G. So the case for G a regular graph, the claim holds. Therefore, the claim holds for all connected graphs that are neither an odd cycle nor a complete graph. □

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Theorem 14.5. The Gallai-Roy Theorem

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Every digraph D contains a directed path with $\chi(D)$ vertices.

Proof. Let k be the number of vertices in a longest directed path of D. Consider a maximal acyclic subgraph D' of D (i.e., D' does not contain a directed cycle and is a subgraph of D with the most arcs which satisfies this property). Because D' is a subgraph of D, each directed path in D' has at most k vertices. We k-colour D by assigning to vertex v the colour $c(v)$, where $c(v) \in \{1, 2, \ldots, k\}$ is the number of vertices of a longest directed path in D' starting at v. We claim this vertex colouring is proper.

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Let (u, v) be an arbitrary arc of D. (1) If (u, v) is an arc of D' then let vPw be a longest directed v-path in D'. If $u \in V(P)$ then there would be a directed cycle in D' (see the figure below), hence $u \notin V(P)$. Thus $uvPw$ is a directed u-path in D' and we have $c(u) > c(v)$.

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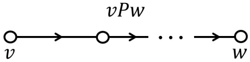
Let (u, v) be an arbitrary arc of D. (1) If (u, v) is an arc of D' then let vPw be a longest directed v-path in D'. If $u \in V(P)$ then there would be a directed cycle in D' (see the figure below), hence $u \notin V(P)$. Thus $uvPw$ is a directed u-path in D' and we have $c(u) > c(v)$.

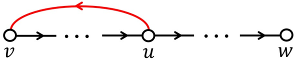
Theorem 14.5 (continued 1)

Proof (continued).

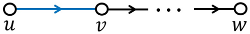
Arc in D'



Longest directed v-path in D'



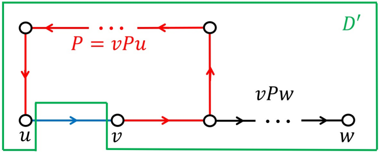
For $u \in V(P)$ then there is a directed cycle in D'.



If $u \notin V(P)$, there is a u-path in D' longer than vPw, implying $c(u) > c(v)$.

(2) If (u, v) is not an arc of D', then $D' + (u, v)$ contains a directed cycle (because D' is maximally acyclic), so D' contains a directed (v, u)-path P.

Arc in D but not in D'

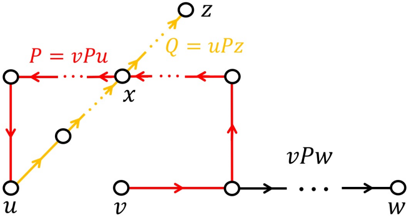


Since D' is maximally acyclic then $D' + (u, v)'$ contains a directed cycle.

Theorem 14.5 (continued 2)

Theorem 14.5. The Gallai-Roy Theorem.
Every digraph D contains a directed path with $\chi(D)$ vertices.

Proof (continued). Let Q be a longest directed u-path in D'. Because D' is acyclic, $V(P) \cap V(Q) = \{u\}$ (for if there were a second vertex x in $V(P) \cap V(Q)$ then there would be a directed cycle in D' from u, along Q to w, then along P back to u; see the figure below).



If $u \neq x \in V(P) \cap V(Q)$ then there is a directed cycle in D'.

Theorem 14.5 (continued 3)

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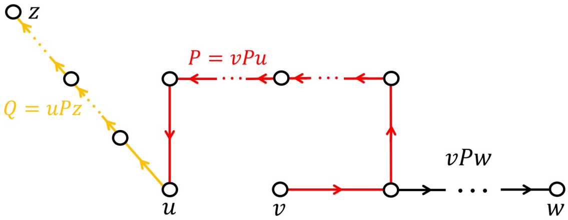
Every digraph D contains a directed path with $\chi(D)$ vertices.
Proof (continued).



Directed v-path $vPuQz = PQ$ in D' is longer than the longest u-path, Q, in D'.

Thus PQ is a directed v path in D' longer than Q (see the figure above), so that $c(v) > c(u)$. In both cases, $c(u) \neq c(v)$. Since (u, v) is an arbitrary arc of D, then the colouring is proper. Since $k \geq \chi(D)$, then D has a directed path of length (at least) $\chi(D)$, as claimed. □