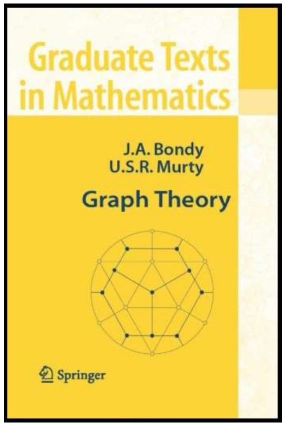
Graph Theory

Chapter 14. Vertex Colourings

14.2. Critical Graphs—Proofs of Theorems



Theorem 14.7

Theorem 14.7. If G is k-critical then $\delta \geq k-1$.

Proof. Let G be k-critical. ASSUME $\delta < k-1$. Let v be a vertex of degree δ in G. Since G is k-critical, then $G-v$ is $(k-1)$-colourable (since we are removing only one vertex then the number of colours required to properly colour $G-v$ is at most one less; since G is k-critical, it is exactly one less). Let $\{V_1, V_2, \ldots, V_{k-1}\}$ be the colour classes of a $(k-1)$-colouring of $G-v$. The vertex v is adjacent to $\delta < k-1$ vertices. So there is some colour class V_j such that v is not adjacent to any vertex of V_j. But then $\{V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{k-1}\}$ is a $(k-1)$-colouring of G, a CONTRADICTION. So the assumption that $\delta < k-1$ is false and hence $\delta \geq j-1$, as claimed. □

Theorem 14.8

Theorem 14.8. No critical graph has a clique cut.

Proof. Let G be a k-critical graph. ASSUME that G has a clique cut S. Denote the S-components of G by $G_1, G_2, \ldots, G_t$. Since G is k-critical, each G_i is $(k-1)$-colourable (though maybe not $(k-1)$-chromatic). Since S is assumed to be a clique, then the vertices of S receive distinct colours in any $(k-1)$-colouring of G_i. Choose the $(k-1)$-colourings of $G_1, G_2, \ldots, G_t$ so that they agree on S. But when these are combined, $G = C_1 \cup G_2 \cup \cdots \cup G_t$, we get a $(k-1)$-colouring of G. But this is a CONTRADICTION because $\chi(G) = k$. So the assumption that G has a clique cut is false, and so critical graph G has no clique cut, as claimed. □

Theorem 14.10

Theorem 14.10. Let G be a k-critical graph with a 2-vertex cut set $\{u, v\}$, and let e be a new edge joining u and v. Then

(1) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of G of Type i for $i \in \{1, 2\}$,

(2) both $H_1 = G_1 + e$ and $H_2 = G_2/\{u, v\}$ are k critical.

Proof. For (1), because G is k-critical then each $\{u, v\}$-component of G is $(k-1)$-colourable. There cannot be $(k-1)$-colourings of the $\{u, v\}$-components all of which agree on $\{u, v\}$, since this would imply a $(k-1)$-colouring of G. Therefore there are two $\{u, v\}$-components G_1 and G_2 such that no $(k-1)$-colouring of G_1 agrees with any $(k-1)$-colouring of G_2. This implies that one component, say G_1, is of Type 1 and the other, say G_2, is of Type 2; notice that if both are Type 1 with u and v colour i in one component and u and v colour j in the other, then the colours i and j can be interchanged in one of the components to produce colourings that agree (and the case of both being Type 2 is similarly resolved).

Theorem 14.10 (continued 1)

Proof (continued). Since G is k-critical then we must have $G = G_1 \cup G_2$, as claimed. (Notice that $G = G_1 \cup G_2$ implies that G does not have an edge joining u and v.)

For (2), because G_1 is of Type 1, $H_1 = G_1 + e$ is k-chromatic (since a $(k-1)$-colouring of G_1 only exists when u and v are the same colour). Let f be any edge of H_1 (we show that $H_1 \setminus f$ is $(k-1)$-colourable, and hence H_1 is k-critical). If $f = e$ then $H_1 \setminus f = H_1 \setminus e = G_1$ and so $H_1 \setminus f$ is $(k-1)$-colourable. Let f be any edge of H_1 other than e (so f is an edge of G_1). Any $(k-1)$-colouring of $G \setminus f$ yields a $(k-1)$-colouring of G_2 and so u and v must receive different colours (since G_2 is Type 2 by (1)). The restriction of such a colouring to G_1 is a $(k-1)$-colouring of $H_1 \setminus f$. So any proper subgraph of H_1 is properly colourable with at most $k-1$ colours and hence H_1 is k-critical, as claimed. We can similarly show that H_2 is k-critical by considering $H_2 \setminus f$ where f is any edge of H_2. □