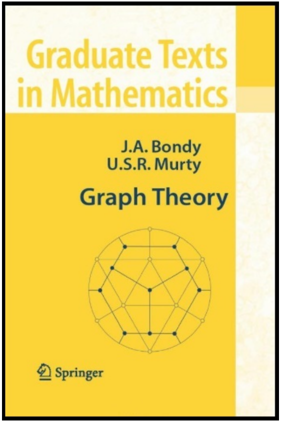
Graph Theory

Chapter 14. Vertex Colourings

14.2. Critical Graphs—Proofs of Theorems



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Proof. Let G be k-critical. ASSUME $\delta < k - 1$. Let v be a vertex of degree δ in G. Since G is k-critical, then $G - v$ is $(k-1)$-colourable (since we are removing only one vertex then the number of colours required to properly colour $G - v$ is at most one less; since G is k-critical, it is exactly one less). Let $\{V_1, V_2, \ldots, V_{k-1}\}$ be the colour classes of a $(k-1)$-colouring of $G - v$.

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Proof. Let G be a k-critical graph. ASSUME that G has a clique cut S. Denote the S-components of G by $G_1, G_2, \ldots, G_t$. Since G is k-critical, each G_i is $(k-1)$-colourable (though maybe not $(k-1)$-chromatic). Since S is assumed to be a clique, then the vertices of S receive distinct colours in any $(k-1)$-colouring of G_i.

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Theorem 14.10

Theorem 14.10. Let G be a k-critical graph with a 2-vertex cut set $\{u, v\}$, and let e be a new edge joining u and v. Then

(1) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of G of Type i for $i \in \{1, 2\}$,

(2) both $H_1 = G_1 + e$ and $H_2 = G_2/\{u, v\}$ are k critical.

Proof. For (1), because G is k-critical then each $\{u, v\}$-component of G is $(k-1)$-colourable. There cannot be $(k-1)$-colourings of the $\{u, v\}$-components all of which agree on $\{u, v\}$, since this would imply a $(k-1)$-colouring of G. Therefore there are two $\{u, v\}$-components G_1 and G_2 such that no $(k-1)$-colouring of G_1 agrees with any $(k-1)$-colouring of G_2.

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For (2), because G_1 is of Type 1, $H_1 = G_1 + e$ is k-chromatic (since a $(k-1)$-colouring of G_1 only exists when u and v are the same colour). Let f be any edge of H_1 (we show that $H_1 \setminus f$ is $(k-1)$-colourable, and hence H_1 is k-critical). If $f = e$ then $H_1 \setminus f = H_1 \setminus e = G_1$ and so $H_1 \setminus f$ is $(k-1)$-colourable. Let f be any edge of H_1 other than e (so f is an edge of G_1). Any $(k-1)$-colouring of $G \setminus f$ yields a $(k-1)$-colouring of G_2 and so u and v must receive different colours (since G_2 is Type 2 by (1)).

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