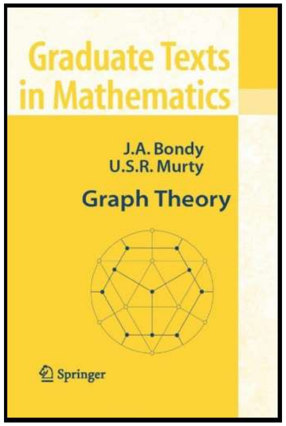
Graph Theory

Chapter 14. Vertex Colourings

14.3. Girth and Chromatic Number—Proofs of Theorems



Theorem 14.11

Theorem 14.11. For each positive integer k, there exists a graph with girth at least k and chromatic number at least k.

Proof. Let positive integer k be given. Recall that $\mathcal{G}_{n,p}$ denotes the probability space of all graphs on n vertices where any two given vertices of a graph are adjacent wih (fixed) probability p (see Section 13.1. Random Graphs). Consider $G \in \mathcal{G}_{n,p}$ and define t as $t = \lceil 2p^{-1} \log n \rceil$. By Theorem 13.6, almost surely the stability number α satisfies $\alpha(t) \leq t$. Let X be the number of cycles of G of length less than k. The expected number of cycles of length i can be computed by first choosing a first vertex, a second vertex, ..., and an ith vertex, which can be done in $n(n-1)(n-2)\cdots(n-i+1)$ ways. Next, we observe that any vertex can act as the "first" vertex of a cycle, so we must divide by the length of the cycle, i. Also, the order of the vertices can be reversed and still yield the same cycle so we also divide by 2.

Theorem 14.11 (continued 1)

Proof (continued). Therefore, the number of possible cycles of length i are

$$\frac{n(n-1)(n-2)\cdots(n-i+1)}{2i} = \frac{(n)_i}{2i} \text{ where } (n)_i \text{ denotes } \frac{n!}{(n-i)!}.$$

Now the probability that all of the necessary i edges are present to form the cycle is p^i. Hence, the expected number of cycles of length i is $\frac{(n)_i}{2i}p^i$. By the linearity of expectation (see equation (13.4) in Section 13.2. Expectation), the expected number of cycles of length less than k is

$$E(X) = \sum_{i=3}^{k-1} \frac{(n)_i}{2i}p^i < \sum_{i=3}^{k-1} \frac{n^i}{1}p^i < \sum_{i=0}^{k}(np)^i = \frac{(np)^k - 1}{(np) - 1},$$

since $(np)^i$ forms a geometric sequence with first term 1 (when $i = 0$), last term $(np)^k$ (when $i = k$), and ratio (np) (recall that the sum of geometric sequence $a_1, a_2, \ldots, a_n$ with ratio r is $a_1(1 - r^n)/(1 - r)$).

Theorem 14.11 (continued 2)

Proof (continued). By Markov's Inequality (Proposition 13.4), $P(X > n/2) < \frac{E(X)}{n/2}$, so that $P(X > n/2) < \frac{E(X)}{n/2} < \frac{2((np)^k - 1)}{n(np-1)}$. If we take $p = n^{-(k-1)/k}$ so that $np = n^1 n^{-(k-1)/k} = n^{1/k}$ and $(np)^k = (n^{1/k})^k = n$ (notice that we are free to choose p to be any value in $[0, 1]$; this just defines the probability space), then we have

$$P(X > n/2) < \frac{2(n-1)}{n(n^{1/k} - 1)} = \frac{2(n-1)}{n^{1+1/k} - n}.$$

So

$$\lim_{n\to\infty} P(X > n/2) \leq \lim_{n\to\infty} \left(\frac{2(n-1)}{2^{1+1/k} - n}\right) = 0.$$

That is, G almost surely has no more than $n/2$ cycles of length less that k. So for n sufficiently large, there exists a graph G on v vertices with stability number at most $t = \lceil 2p^{-1} \log n \rceil$ and no more than $n/2$ cycles of length less than k.

Theorem 14.11 (continued 3)

Theorem 14.11. For each positive integer k, there exists a graph with girth at least k and chromatic number at least k.

Proof (continued). We now modify this graph G. We delete one vertex of G from each cycle of length less than k. This means that at most $n/2$ vertices are deleted, yielding a graph G' on at least $n/2$ vertices with girth at least k. Recall that the stability number of a graph is the size of a largest stable set (or "independent set"). By deleting vertices from graph G, we create graph G' with a smaller stability number (deleting a vertex and all edges incident to it can only delete a vertex from some stable set and cannot add any vertices to a stable set of G), so that $\alpha(G') \le \alpha(G)$. So $\chi(G') \le \chi(G) \le t$. Since $\chi(G') \ge v(G')/\alpha(G')$ (see equation (14.1) of Section 14.1. Chromatic Number), then $\chi(G') \ge \dfrac{v(G')}{\alpha(G')} \ge \dfrac{n/2}{t}$.

Theorem 14.11 (continued 4)

Proof (continued). Now

$$\chi(G') \ge \frac{n}{2t} = \frac{n}{2\lceil 2p^{-1}\log n\rceil} \ge \frac{n}{2(2p^{-1}\log n + 1)}$$

and

$$\lim_{n\to\infty} \frac{\left(\frac{n}{2(2p^{-1}\log n+1)}\right)}{\left(\frac{n^{1/k}}{8\log n}\right)} = \lim_{n\to\infty} \frac{8n\log n}{2n^{1/k}(2p^{-1}\log n+1)}$$

$$= \lim_{n\to\infty} \frac{4n^{1-1/k}\log n}{2p^{-1}\log n + 1} = \lim_{n\to\infty} \frac{4n^{1-1/k}}{2p^{-1} + 1/\log n} = \infty.$$

So $\chi(G')$ can be made as large as desired by making n sufficiently large (to describe the infinite limit informally). In particular, for any positive integer k, there is a graph G' such that $\chi(G') \ge k$ and the girth of G' is at least k, as claimed. □

Theorem 14.12

Theorem 14.12. For any positive integer k, there exists a triangle-free k-chromatic graph.

Proof. For $k = 1$ and $k = 2$, the graphs K_1 and K_2 have the required property. We use these as base cases in an induction proof based on the value of k. For the induction step, suppose that a triangle-free graph G_k with chromatic number $k \ge 2$ exists. Let the vertices of G_k be $v_1, v_2, \ldots, v_n$. Form the graph G_{k+1} from G_k as: add $n+1$ new vertices $u_1, u_2, \ldots, u_n, v$, and then for $1 \le i \le n$, join u_i to the neighbors of v_i in G_k and also join u_i to v. Notice that $u_1, u_2, \ldots, u_n$ is a stable set in G_{k+1}.

As an example, if $G_2 = K_2$ with vertices v_1 and v_2, then G_3 has the new vertices u_1, u_2, v with u_1 adjacent to v_2 and v, and u_2 adjacent to v_1 and v to give G_3 as a 5-cycle (see Figure 14.6 left, where the labels v_3, v_5, v_4 should be replaced with the labels u_1, u_2, v, respectively).

Theorem 14.12 (continued 1)

Proof (continued). Also, for G_4 we label the vertices of G_3 as v_1, v_2, v_3, v_4, v_5 and add the new vertices $u_1, u_2, u_3, u_4, u_5, v$ with u_i adjacent to the neighbors of v_i in G_3 and also adjacent to v. This gives the graph in Figure 14.6 right.

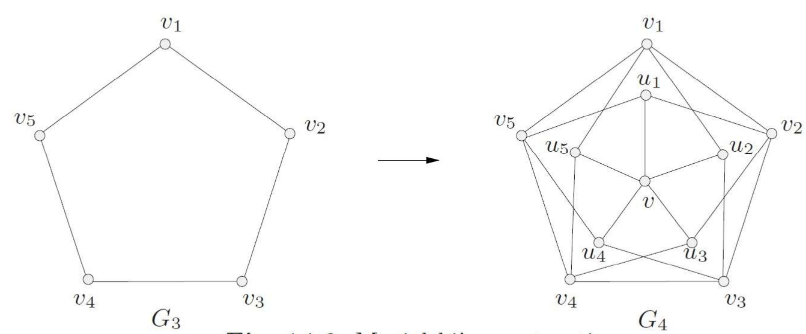


Fig. 14.6. Mycielski's construction

Theorem 14.12 (continued 2)

Proof (continued). We claim that G_{k+1} is triangle-free. Since $u_1, u_2, \ldots, u_n$ is a stable set in G_{k+1}, then no triangle can contain more than one u_i and since v is only adjacent to u_i's then it cannot be in a triangle. If $u_i v_j v_k u_i$ were a triangle in G_{k+1} then $v_i v_j v_k v_i$ would be a triangle in G_k (since u_i is adjacent to the neighbors of v_i). But this is a triangle in G_k, contradicting to the induction hypothesis. So G_{k+1} is triangle-free, as claimed.

We claim G_{k+1} is $(k+1)$-chromatic. First, G_{k+1} is $(k+1)$-colourable because G_k is k-colourable by the induction hypothesis and vertex u_i can be assigned the same colour as v_i (since u_i and v_i are not adjacent, but the neighbors of v_i are also neighbors of u_i). Then v can be assigned a new, $(k+1)$-st, colour. Second, ASSUME G_{k+1} is k-colourable. The colouring restricted to the vertices $\{v_1, v_2, \ldots, v_n\}$ of G_k is a k-colouring of k-chromatic G_k. By Exercise 14.1.3(a), for each colour j there is a vertex v_i of colour j which is adjacent in G_k to vertices of every other colour.

Theorem 14.12 (continued 3)

Theorem 14.12. For any positive integer k, there exists a triangle-free k-chromatic graph.

Proof (continued). Since u_i has precisely the same neighbors in G_{k+1} which are vertices of G_k as v_i has in G_k, then vertex u_i must also have colour j. So each of the k colours appears on at least one of the vertices u_i. But vertex v is adjacent to all of the u_i and so it cannot be assigned any of the k colours in a proper colouring of G_{k+1}, a CONTRADICTION. So the assumption that G_{k+1} is k-colourable is false. Therefore, G_{k+1} is triangle-free and has chromatic number$k+1$. This establishes the induction step. Therefore, by induction, the claim holds for all $k \in \mathbb{N}$, as needed. □