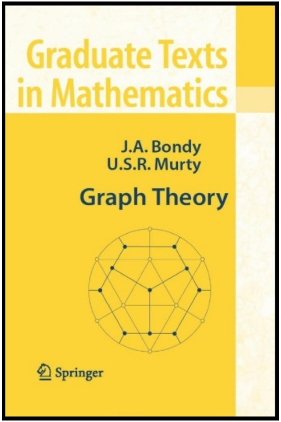
Graph Theory

Chapter 14. Vertex Colourings

14.3. Girth and Chromatic Number—Proofs of Theorems



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Proof. Let positive integer k be given. Recall that $\mathcal{G}_{n,p}$ denotes the probability space of all graphs on n vertices where any two given vertices of a graph are adjacent wih (fixed) probability p (see Section 13.1. Random Graphs). Consider $G \in \mathcal{G}_{n,p}$ and define t as $t = \lceil 2p^{-1} \log n \rceil$. By Theorem 13.6, almost surely the stability number α satisfies $\alpha(t) \leq t$.

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Theorem 14.11 (continued 1)

Proof (continued). Therefore, the number of possible cycles of length i are

$$\frac{n(n-1)(n-2)\cdots(n-i+1)}{2i} = \frac{(n)_i}{2i} \text{ where } (n)_i \text{ denotes } \frac{n!}{(n-i)!}.$$

Now the probability that all of the necessary i edges are present to form the cycle is p^i. Hence, the expected number of cycles of length i is $\frac{(n)_i}{2i}p^i$. By the linearity of expectation (see equation (13.4) in Section 13.2. Expectation), the expected number of cycles of length less than k is

$$E(X) = \sum_{i=3}^{k-1} \frac{(n)_i}{2i} p^i < \sum_{i=3}^{k-1} \frac{n^i}{1} p^i < \sum_{i=0}^{k} (np)^i = \frac{(np)^k - 1}{(np) - 1},$$

since $(np)^i$ forms a geometric sequence with first term 1 (when $i = 0$), last term $(np)^k$ (when $i = k$), and ratio (np) (recall that the sum of geometric sequence $a_1, a_2, \ldots, a_n$ with ratio r is $a_1(1 - r^n)/(1 - r)$).

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Theorem 14.11 (continued 2)

Proof (continued). By Markov's Inequality (Proposition 13.4), $P(X > n/2) < \frac{E(X)}{n/2}$, so that $P(X > n/2) < \frac{E(X)}{n/2} < \frac{2((np)^k - 1)}{n(np-1)}$. If we take $p = n^{-(k-1)/k}$ so that $np = n^1 n^{-(k-1)/k} = n^{1/k}$ and $(np)^k = (n^{1/k})^k = n$ (notice that we are free to choose p to be any value in $[0,1]$; this just defines the probability space), then we have

$$P(X > n/2) < \frac{2(n-1)}{n(n^{1/k}-1)} = \frac{2(n-1)}{n^{1+1/k} - n}.$$

So

$$\lim_{n\to\infty} P(X > n/2) \leq \lim_{n\to\infty} \left(\frac{2(n-1)}{2^{1+1/k} - n} \right) = 0.$$

That is, G almost surely has no more than $n/2$ cycles of length less that k. So for n sufficiently large, there exists a graph G on v vertices with stability number at most $t = \lceil 2p^{-1} \log n \rceil$ and no more than $n/2$ cycles of length less than k.

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Theorem 14.11 (continued 3)

Theorem 14.11. For each positive integer k, there exists a graph with girth at least k and chromatic number at least k.

Proof (continued). We now modify this graph G. We delete one vertex of G from each cycle of length less than k. This means that at most $n/2$ vertices are deleted, yielding a graph G' on at least $n/2$ vertices with girth at least k. Recall that the stability number of a graph is the size of a largest stable set (or "independent set"). By deleting vertices from graph G, we create graph G' with a smaller stability number (deleting a vertex and all edges incident to it can only delete a vertex from some stable set and cannot add any vertices to a stable set of G), so that $\alpha(G') \leq \alpha(G)$. So $\chi(G') \leq \chi(G) \leq t$. Since $\chi(G') \geq v(G')/\alpha(G')$ (see equation (14.1) of Section 14.1. Chromatic Number), then $\chi(G') \geq \dfrac{v(G')}{\alpha(G')} \geq \dfrac{n/2}{t}$.

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Theorem 14.11 (continued 4)

Proof (continued). Now

$$\chi(G') \geq \frac{n}{2t} = \frac{n}{2\lceil 2p^{-1}\log n\rceil} \geq \frac{n}{2(2p^{-1}\log n + 1)}$$

and

$$\lim_{n\to\infty} \frac{\left(\frac{n}{2(2p^{-1}\log n+1}\right)}{\left(\frac{n^{1/k}}{8\log n}\right)} = \lim_{n\to\infty} \frac{8n\log n}{2n^{1/k}(2p^{-1}\log n + 1)}$$

$$= \lim_{n\to\infty} \frac{4n^{1-1/k}\log n}{2p^{-1}\log n + 1} = \lim_{n\to\infty} \frac{4n^{1-1/k}}{2p^{-1} + 1/\log n} = \infty.$$

So $\chi(G')$ can be made as large as desired by making n sufficiently large (to describe the infinite limit informally). In particular, for any positive integer k, there is a graph G' such that $\chi(G') \geq k$ and the girth of G' is at least k, as claimed. □

Theorem 14.12

Theorem 14.12. For any positive integer k, there exists a triangle-free k-chromatic graph.

Proof. For $k = 1$ and $k = 2$, the graphs K_1 and K_2 have the required property. We use these as base cases in an induction proof based on the value of k. For the induction step, suppose that a triangle-free graph G_k with chromatic number $k \geq 2$ exists. Let the vertices of G_k be $v_1, v_2, \ldots, v_n$. Form the graph G_{k+1} from G_k as: add $n + 1$ new vertices $u_1, u_2, \ldots, u_n, v$, and then for $1 \leq i \leq n$, join u_i to the neighbors of v_i in G_k and also join u_i to v. Notice that $u_1, u_2, \ldots, u_n$ is a stable set in G_{k+1}.

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As an example, if $G_2 = K_2$ with vertices v_1 and v_2, then G_3 has the new vertices u_1, u_2, v with u_1 adjacent to v_2 and v, and u_2 adjacent to v_1 and v to give G_3 as a 5-cycle (see Figure 14.6 left, where the labels v_3, v_5, v_4 should be replaced with the labels u_1, u_2, v, respectively).

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Theorem 14.12 (continued 1)

Proof (continued). Also, for G_4 we label the vertices of G_3 as v_1, v_2, v_3, v_4, v_5 and add the new vertices $u_1, u_2, u_3, u_4, u_5, v$ with u_i adjacent to the neighbors of v_i in G_3 and also adjacent to v. This gives the graph in Figure 14.6 right.

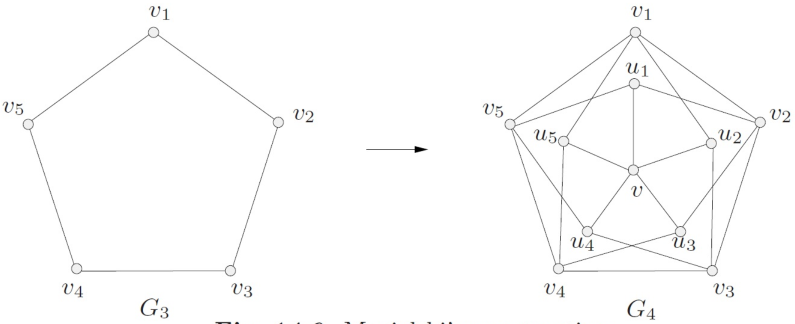

Fig. 14.6. Mycielski's construction

Theorem 14.12 (continued 2)

Proof (continued). We claim that G_{k+1} is triangle-free. Since $u_1, u_2, \ldots, u_n$ is a stable set in G_{k+1}, then no triangle can contain more than one u_i and since v is only adjacent to u_i's then it cannot be in a triangle. If $u_i v_j v_k u_i$ were a triangle in G_{k+1} then $v_i v_j v_k v_i$ would be a triangle in G_k (since u_i is adjacent to the neighbors of v_i). But this is a triangle in G_k, contradicting to the induction hypothesis. So G_{k+1} is triangle-free, as claimed.

We claim G_{k+1} is $(k+1)$-chromatic. First, G_{k+1} is $(k+1)$-colourable because G_k is k-colourable by the induction hypothesis and vertex u_i can be assigned the same colour as v_i (since u_i and v_i are not adjacent, but the neighbors of v_i are also neighbors of u_i). Then v can be assigned a new, $(k+1)$-st, colour. Second, ASSUME G_{k+1} is k-colourable. The colouring restricted to the vertices $\{v_1, v_2, \ldots, v_n\}$ of G_k is a k-colouring of k-chromatic G_k. By Exercise 14.1.3(a), for each colour j there is a vertex v_i of colour j which is adjacent in G_k to vertices of every other colour.

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Theorem 14.12 (continued 3)

Theorem 14.12. For any positive integer k, there exists a triangle-free k-chromatic graph.

Proof (continued). Since u_i has precisely the same neighbors in G_{k+1} which are vertices of G_k as v_i has in G_k, then vertex u_i must also have colour j. So each of the k colours appears on at least one of the vertices u_i. But vertex v is adjacent to all of the u_i and so it cannot be assigned any of the k colours in a proper colouring of G_{k+1}, a CONTRADICTION. So the assumption that G_{k+1} is k-colourable is false. Therefore, G_{k+1} is triangle-free and has chromatic number$k + 1$. This establishes the induction step. Therefore, by induction, the claim holds for all $k \in \mathbb{N}$, as needed. □