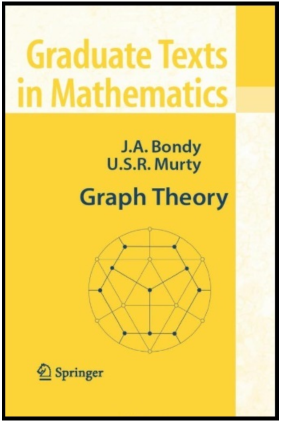
Graph Theory

Chapter 14. Vertex Colourings

14.4. Perfect Graphs—Proofs of Theorems



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Proposition 14.15

Proposition 14.15. Let S be a stable set in a minimally imperfect graph G. Then $\omega(G - S) = \omega(G)$.

Proof. This is easy with the use of Exercise 14.4.5, which states that for S a stable set in a minimally imperfect graph G we have

$$\omega(G - S) \leq \omega(G) \leq \chi(G) - 1 \leq \chi(G - S) = \omega(G - S).$$

Since the left and right terms are the same, then the inequalities reduce to equalities. □

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Lemma 14.16

Lemma 14.16. Let G be a minimally imperfect graph with stability number α and clique number ω. Then G contains $\alpha\omega + 1$ stable sets $S_0, S_1, \ldots, S_{\alpha\omega}$ and $\alpha\omega + 1$ cliques $C_0, C_1, \ldots, C_{\alpha\omega}$ such that:

- each vertex of G belongs to precisely α of the stable sets S_i,
- each clique C_i has ω vertices,
- $C_i \cap S_i = \varnothing$ for $0 \leq i \leq \alpha\omega$, and
- $|C_i \cap S_j| = 1$ for $0 \leq i < j \leq \alpha\omega$.

Proof. Let S_0 be a stable set of α vertices of G, and let $v \in S_0$ (of course $|S_0| \geq 1$). The graph $G - v$ is perfect because G is minimally imperfect (note that $G - v$ is the subgraph of G induced by $V(G) \setminus \{v\}$). So by the definition of "perfect graph" and Note 14.1.B,
$\chi(G - v) = \omega(G - v) \leq \omega(G)$.

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$\chi(G - v) = \omega(G - v) \leq \omega(G)$. Since the colours classes of a proper colouring are each stable sets, we then have for any $v \in S_0$ that the set $V \setminus \{v\}$ can be partitioned into a family $\mathcal{S}_v$ of ω stable sets (the inequality tells us that there are *at most* ω stable sets, but the stable sets can be further subdivided to get a total of ω stable sets).

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Lemma 14.16 (continued 1)

Proof (continued). Since $|S_0| = \alpha$ then there are α such v in S_0, since each family $\mathcal{S}_v$ contains ω stable sets, then there are $\alpha\omega$ stable sets in the totality of the families $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\alpha$. Denote these stable sets as $S_1, S_2, \ldots, S_{\alpha\omega}$. We'll see below that each S_j is nonempty when we establish the fourth claim. For each v a vertex of G where $v \in S_0$, we have that $v \in S_i$ for some unique S_i in family $\mathcal{S}_w$ where $w \neq v$. So this v is in S_0 and all but one of $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\alpha$; that is, v is in α of the stable sets S_i. For each v a vertex of G where $v \notin S_0$, we have that v is in some unique S_i in family $\mathcal{S}_j$ for each $1 \leq j \leq \alpha$; that is, v is in α of the stable sets S_i. Therefore the first claim holds.

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By Proposition 14.15, the clique numbers satisfy $\omega(G - S_i) = \omega(G)$ for each $0 \leq i \leq \alpha\omega$. Therefore there is a maximum clique C_i of $G - S_i$ that is also a maximum clique of G. Each C_i has $\omega = \omega(G)$ vertices, so the second claim holds. So there is a maximum clique C_i of G that is disjoint from S_i and the third claim holds.

Lemma 14.16 (continued 1)

Proof (continued). Since $|S_0| = \alpha$ then there are α such v in S_0, since each family $\mathcal{S}_v$ contains ω stable sets, then there are $\alpha\omega$ stable sets in the totality of the families $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\alpha$. Denote these stable sets as $S_1, S_2, \ldots, S_{\alpha\omega}$. We'll see below that each S_j is nonempty when we establish the fourth claim. For each v a vertex of G where $v \in S_0$, we have that $v \in S_i$ for some unique S_i in family $\mathcal{S}_w$ where $w \neq v$. So this v is in S_0 and all but one of $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\alpha$; that is, v is in α of the stable sets S_i. For each v a vertex of G where $v \notin S_0$, we have that v is in some unique S_i in family $\mathcal{S}_j$ for each $1 \leq j \leq \alpha$; that is, v is in α of the stable sets S_i. Therefore the first claim holds.

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Lemma 14.16 (continued 2)

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- $C_i \cap S_i = \varnothing$ for $0 \le i \le \alpha\omega$, and
- $|C_i \cap S_j| = 1$ for $0 \le i < j \le \alpha\omega$.

Proof (continued). Because each of the ω vertices of C_i lie in α of the stable sets (by the first claim), there are $\alpha\omega + 1$ stable sets (but $C_i \cap S_i = \varnothing$ for $0 \le i \le \alpha\omega$ by the third claim), and no two vertices of clique C_i can belong to a common stable set (since vertices of a clique are adjacent and vertices of a stable set are not adjacent), then each C_i shares exactly one point with each S_j where $i \neq j$. That is, $|C_i \cap S_j| = 1$ for $0 \le i < j \le \alpha\omega$, and the fourth claim holds. □

Theorem 14.14

Theorem 14.14. A graph G is perfect if and only if every induced subgraph H of G satisfies the inequality $v(H) \leq \alpha(H)\omega(H)$.

Proof. Suppose that G is a perfect graph so that $\chi(H) = \omega(H)$ for all induced subgraphs H of G. Let H be any (fixed) induced subgraph of G. Then H is $\omega(H)$-colurable (since $\chi(H) = \omega(H)$). Since a colour class is an independent set and the largest independent set is of size α, then the number of vertices satisfies $v(H) \leq \alpha(H)\chi(H) = \alpha(H)\omega(H)$. So if G is perfect, then for every induced subgraph H of G we have $v(H) \leq \alpha(H)\omega(H)$.

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We now prove the converse (actually, the contrapositive of the converse). Suppose G is not perfect. We want to show that $v(G) \geq \alpha(G)\omega(G) + 1$. If we can show this for minimally imperfect graphs, then it will hold for all imperfect graphs. So without loss of generality, suppose G is minimally imperfect.

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Theorem 14.14 (continued 1)

Proof (continued). Consider the stable sets S_i for $0 \leq i \leq \alpha\omega$ and the cliques C_i for $0 \leq i \leq \alpha\omega$ as given for a minimally imperfect graph in Lemma 14.16. Let $\mathbf{S}$ and $\mathbf{C}$ be the $n \times (\alpha\omega + 1)$ incidence matrices of the families. Now the transpose of $\mathbf{S}$, $\mathbf{S}^t$, is $(\alpha\omega + 1) \times n$ and so the product $\mathbf{S}^t\mathbf{C}$ exists and is $(\alpha\omega + 1) \times (\alpha\omega + 1)$. Now the ith row of $\mathbf{S}^t$ consists of 0's and 1's, where the kth entry is 1 if $k \in S_i$ and 0 if $k \notin S_i$. The jth column of $\mathbf{C}$ consists of 0's and 1's where the kth entry is 1 if $k \in C_i$ and 0 if $k \notin C_i$. The (i, j)-entry of $\mathbf{S}^t\mathbf{C}$ is the inner product (or "dot product") of the ith row of $\mathbf{S}^t$ and the jth column of $\mathbf{C}$. By the third claim of Lemma 14.16 we have $C_i \cap S_i = \varnothing$ for $0 \leq i \leq \alpha\omega$, so in the ith row of $\mathbf{S}^t$ and the ith column of $\mathbf{C}$, if the kth entry of one of these is 1 then the kth entry of the other is 0. So the inner product is 0 and the (i, i)-entry of $\mathbf{S}^t\mathbf{C}$ is 0. By the fourth claim of Lemma 14.16 we have $|C_i \cap S_j| = 1$ for $0 \leq i < j \leq \alpha\omega$, so in the ith row of $\mathbf{S}^t$ and the jth column of $\mathbf{C}$ the k entry of both is 1 only once. So the inner product is 1 and the (i, j)-entry of $\mathbf{S}^t\mathbf{C}$ is 1 for $i \neq j$.

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Theorem 14.14 (continued 2)

Proof (continued). Therefore, $\mathbf{S}^t\mathbf{C} = \mathbf{J} - \mathbf{I}$ where $\mathbf{J}$ is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix of all 1's and $\mathbf{I}$ is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ identity matrix. Now $\mathbf{J} - \mathbf{I}$ is nonsingular (i.e., invertible) because the inverse is $(1/(\alpha\omega))\mathbf{J} - \mathbf{I}$ because

$$(\mathbf{J} - \mathbf{I})\left(\frac{1}{\alpha\omega}\mathbf{J} - \mathbf{I}\right) = \frac{1}{\alpha\omega}\mathbf{J}^2 - \mathbf{J} - \frac{1}{\alpha\omega}\mathbf{J} + \mathbf{I}^2 = \frac{1}{\alpha\omega}\mathbf{J}^2 - \left(1 + \frac{1}{\alpha\omega}\right)\mathbf{J} + \mathbf{I}$$

$$= \frac{1}{\alpha\omega}((\alpha\omega + 1)\mathbf{J}) - \left(\frac{\alpha\omega + 1}{\alpha\omega}\right)\mathbf{J} + \mathbf{I} = \frac{\alpha\omega + 1}{\alpha\omega}\mathbf{J} - \frac{\alpha\omega + 1}{\alpha\omega}\mathbf{J} + \mathbf{I} = \mathbf{I}.$$

This shows that $(1/(\alpha\omega))\mathbf{J} - \mathbf{I}$ is a right inverse of $\mathbf{J} - \mathbf{I}$, but since $\mathbf{I}$ and $\mathbf{J}$ commute, this is sufficient to establish that we have a two-sided inverse (see also "Theorem 1.11. A Commutative Property" in my Linear Algebra [MATH 2010] online notes on Section 1.5. Inverses of Matrices, and Linear Systems). Since $\mathbf{J} - \mathbf{I}$ is invertible, then it must be a full-rank $\alpha\omega + 1$ (see "Theorem 2.6. An Invertible Criteria" in my Linear Algebra online notes on Section 2.2. The Rank of a Matrix).

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$$(\mathbf{J} - \mathbf{I})\left(\frac{1}{\alpha\omega}\mathbf{J} - \mathbf{I}\right) = \frac{1}{\alpha\omega}\mathbf{J}^2 - \mathbf{J} - \frac{1}{\alpha\omega}\mathbf{J} + \mathbf{I}^2 = \frac{1}{\alpha\omega}\mathbf{J}^2 - \left(1 + \frac{1}{\alpha\omega}\right)\mathbf{J} + \mathbf{I}$$

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Theorem 14.14 (continued 3)

Theorem 14.14. A graph G is perfect if and only if every induced subgraph H of G satisfies the inequality $v(H) \leq \alpha(H)\omega(H)$.

Proof (continued). Recall that the rank of a matrix in the (common) dimension of the row space and the column space. In general, $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$ (see Exercises 2.2.18 and 2.2.20 in J. Fraleigh and R. Beauregard's *Linear Algebra*, 3rd Edition, Addison-Wesley (1994), or see Theorem 3.3.5 in my online notes for Theory of Matrices [MATH 5090] on Section 3.3. Matrix Rank and the Inverse of a Full Rank Matrix) so

$$\alpha\omega + 1 = \text{rank}(\mathbf{J} - \mathbf{I}) = \text{rank}(\mathbf{S}^t\mathbf{C}) \leq \min\{\text{rank}(\mathbf{S}^t), \text{rank}(\mathbf{C})\}.$$

Now the row rank equals the column rank for any matrix (see "Theorem 2.4. Row Rank Equals Column Rank" in my Linear Algebra notes on Section 2.2. The Rank of a Matrix), so we have $\text{rank}(\mathbf{S}) = \alpha\omega + 1$ and $\text{rank}(\mathbf{C}) = \alpha\omega + 1$ since the rank can't be greater than the number of rows or the number of columns.

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Theorem 14.14 (continued 4)

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Proof (continued).
Both **S** and **C** have n row, so the number of rows must be at least as big as the rank, hence $n \geq \alpha\omega + 1$. That is, for minimally imperfect graph G, $v(G) \geq \alpha(G)\omega(G) + 1$. As explained above, the general claim now holds. □