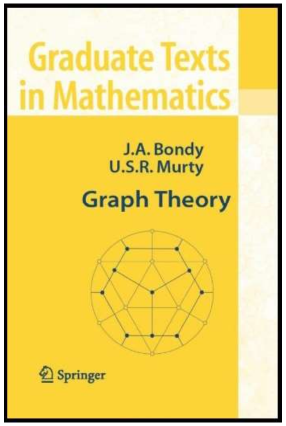
Graph Theory

Chapter 14. Vertex Colourings

14.5. List Colourings—Proofs of Theorems



Proposition 14.20

Theorem 14.20. Let $D = (V, A)$ be a digraph each of whose induced sub-digraphs has a kernel. For $v \in V$, let $L(v)$ be an arbitrary list of at least $d^+(v) + 1$ colours. Then D admits an L-colouring.

Proof. We give a proof on the number n of vertices of D. With $n = 1$, D has 1 vertex and no arcs (we still consider simple, loopless graphs and digraphs in this chapter), so the base case holds trivially. For the induction hypothesis, suppose the result holds for all digraphs on $\not\leq k$ vertices. Let D be a digraph on $n = k + 1$ vertices each of whose induced sub-digraphs has a kernel and for $v \in V$, $L(v)$ has at least $d^+(v) + 1$ colours. Let V_1 be the set of vertices of D whose lists include colour 1 (we can assume $V_1 \neq \varnothing$, or else we could rename the colours so that this holds). By hypothesis, the induced digraph $D[V_1]$ has a kernel $S_1 \subseteq V_1$. Colour the vertices of S_1 with colour 1, and set $D' = D - S_1$ and $L'(v) = (v) \setminus \{1\}$ for $v \in V(D')$.

Proposition 14.20 (continued)

Proof (continued). For any vertex v of D' whose list $L(v)$ did not contain colour 1 originally,

$$\begin{aligned} |L'(v)| &= |L(v)| \geq d^+_D(v) + 1 \text{ by hypothesis} \\ &\geq d^+_{D'}(v) + 1 \text{ since } V(D) \supseteq V(D - S_1) = V(D'). \end{aligned}$$

For any vertex v of D' whose list $L(v)$ did contain colour 1 originally,

$$|L'(v)| = |L(v)| - 1 \geq d^+_D(v) \text{ by hypothesis.}$$

Now for v of D' whose list did not originally contain colour 1, v dominates some vertex of kernel S_1 (by the definition of "kernel"); so its out-degree in $D' = D - S_1$ is smaller than in D. That is $d^+_D(v) \geq d^+_{D'}(v) + 1$. Hence, $|L'(v)| \geq d^+_D(v) \geq d^+_{D'}(v) + 1$. Since D' has less than or equal to k vertices, by the induction hypothesis D' has an L'-colouring. This L'-colouring of D', along with the colouring of the vertices of S_1 with colour 1, gives an L-colouring of D. So by induction, the claim holds for all graphs D satisfying the hypothesis. □