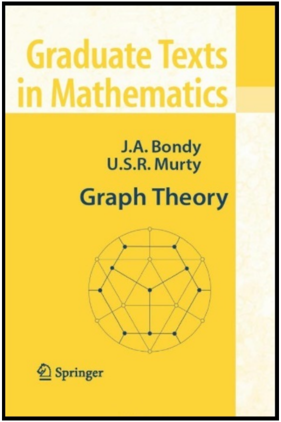
Graph Theory

Chapter 14. Vertex Colourings

14.5. List Colourings—Proofs of Theorems



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Theorem 14.20. Let $D = (V, A)$ be a digraph each of whose induced sub-digraphs has a kernel. For $v \in V$, let $L(v)$ be an arbitrary list of at least $d^+(v) + 1$ colours. Then D admits an L-colouring.

Proof. We give a proof on the number n of vertices of D. With $n = 1$, D has 1 vertex and no arcs (we still consider simple, loopless graphs and digraphs in this chapter), so the base case holds trivially. For the induction hypothesis, suppose the result holds for all digraphs on $\not\leq k$ vertices. Let D be a digraph on $n = k + 1$ vertices each of whose induced sub-digraphs has a kernel and for $v \in V$, $L(v)$ has at least $d^+(v) + 1$ colours.

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Proposition 14.20 (continued)

Proof (continued). For any vertex v of D' whose list $L(v)$ did not contain colour 1 originally,

$$\begin{aligned} |L'(v)| &= |L(v)| \geq d_D^+(v) + 1 \text{ by hypothesis} \\ &\geq d_{D'}^+(v) + 1 \text{ since } V(D) \supseteq V(D - S_1) = V(D'). \end{aligned}$$

For any vertex v of D' whose list $L(v)$ did contain colour 1 originally,

$$|L'(v)| = |L(v)| - 1 \geq d_D^+(v) \text{ by hypothesis.}$$

Now for v of D' whose list did not originally contain colour 1, v dominates some vertex of kernel S_1 (by the definition of "kernel"); so its out-degree in $D' = D - S_1$ is smaller than in D. That is $d_D^+(v) \geq d_{D'}^+(v) + 1$. Hence, $|L'(v)| \geq d_D^+(v) \geq d_{D'}^+(v) + 1$. Since D' has less than or equal to k vertices, by the induction hypothesis D' has an L'-colouring. This L'-colouring of D', along with the colouring of the vertices of S_1 with colour 1, gives an L-colouring of D. So by induction, the claim holds for all graphs D satisfying the hypothesis. □

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