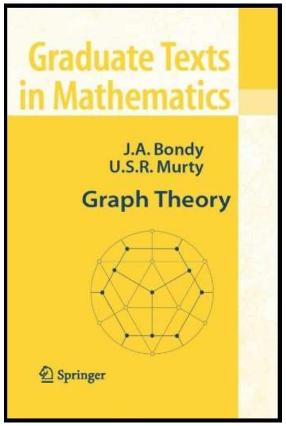
Graph Theory

Chapter 14. Vertex Colourings

14.6. The Adjacency Polynomial—Proofs of Theorems



Proposition 14.23

Proposition 14.23. Let f be a polynomial, not the zero polynomial, over a field $\mathbb{F}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, of degree d_i in x_i for $1 \le i \le n$. Let L_i be a set of $d_i + 1$ elements of $\mathbb{F}$ for $1 \le i \le n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \ne 0$.

Proof. We give an inductive proof based on the number n of variables in f. For the base case $n = 1$, we have a polynomial in one variable and we know that such a polynomial has at most n distinct roots; see Corollary 23.5 in my online notes for Introduction to Modern Algebra (MATH 4127/5127) on Section IV.23. Factorizations of Polynomials. Since L_1 contains $d_1 + 1 = n + 1$ elements of $\mathbb{F}$, then for some $t \in L_1$ we must have $f(t) \ne 0$ so that the base case holds. For the induction hypothesis, suppose the claim holds for all polynomials in $n = k - 1$ variables. Suppose f is a polynomial (not the zero polynomial) in $n = k$ variables where $n \ge 2$.

Proposition 14.23 (continued)

Proof (continued). First, express f in the form $f = \sum_{j=0}^{d_n} f_j x_n^j$ where the coefficients f_j for $0 \le j \le n$ are themselves polynomials in the $n - 1 = k - 1$ variables $x_1, x_2, \ldots, x_{n-1}$. Since f is not the zero polynomial by hypothesis, then f_j is not the zero polynomial for some $o \le j \le d_n$. By the induction hypothesis, there is $t_i \in L_i$ for $1 \le i \le k - 1$ such that $f_j(t_1, t_2, \ldots, t_{k-1}) \ne 0$. Therefore the polynomial $\sum_{j=0}^{d_k} f_j(t_1, t_2, \ldots, t_{k-1}) x_k^j$ is not the zero polynomial. This is a polynomial in one variable and so by the case $n = 1$, for some $t_k \in L_k$ we have $\sum_{j=0}^{d_k} f_j(t_1, t_2, \ldots, t_{k-1})(t_k)^j \ne 0$. That is, $f(t_1, t_2, \ldots, t_k) \ne 0$. So the induction step holds. Therefore by mathematical induction, the result holds in general. □

Theorem 14.24. The Combinatorial Nullstellensatz

Theorem 14.24. The Combinatorial Nullstellensatz. Let f be a polynomial over a field $\mathbb{F}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Suppose that the total degree of f is $\sum_{i=1}^n d_i$ and that the coefficient in f of $\prod_{i=1}^n x_i^{d_i}$ is nonzero. Let L_i be a set of $d_i + 1$ elements of $\mathbb{F}$ for $1 \le i \le n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \ne 0$.

Proof. For $1 \le i \le n$, set $f_i = \prod_{t \in L_i}(x_i - t)$ (notice that i is fixed and it is $t \in L_i$ that varies). The f_i is a polynomial of degree $|L_i| = d_i + 1$ with leading term $x_i^{d_i+1}$. Hence we can $f_i = g_i + x_i^{d_i+1}$ where g_i is a polynomial in X_i of degree at most d_i. In polynomial f, we can substitute for all expressions $x_i^{d_i+1}$ the expression $-g_i$ where the substitution is repeated until all powers of x_i are less than $d_i + 1$, to obtain a "new polynomial" in which the degree of x_i does not exceed d_i. We can perform similar substitutions in the new polynomial for all other i with $1 \le i \le n$ to create the polynomial g in which each x_i has an exponent of at most d_i (for all $1 \le i \le n$).

Theorem 14.24 (continued 1)

Proof (continued). Moreover, since for each $t \in L_i$ we have $f_i(t) = 0$ and since $f_i = g_i + x_i^{d_i+1}$, then $t^{d_i+1} = -g_i)t)$ for all $t \in L_i$ where $1 \leq i \leq n$. This implies that $g(\mathbf{t}) = f(\mathbf{t})$ for all $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ because for the components of $\mathbf{t}$ the substitution $t^{d_i+1} = -g_i(t)$ is valid.

Now for every monomial in f, the sum of the exponents is at most $\sum_{i=1}^n d_i$, so if a monomial has x_i to a power greater than or equal to $d_i + 1$ then one of the other terms of the monomial must be of the form $x_j^{e_j}$ where $e_j < d_j$. Hence, the process above of reducing the exponent on x_i to be at most d_i to create g yields a monomial of total degree strictly less than $\sum_{i=1}^n d_i$. Notice that g has the monomial term $\prod_{i=1}^n x_i^{d_i}$ (with some coefficient) is present in both f and g (since the process of creating g does not affect the $x_i^{d_i}$ terms). Since all other monomials of g are of total degree strictly less than $\sum_{i=1}^n d_i$, then they cannot cancel with $\prod_{i=1}^n x_i^{d_i}$. That is, g is not the zero polynomial.

Theorem 14.24 (continued 2)

Theorem 14.24. The Combinatorial Nullstellensatz.
Let f be a polynomial over a field $\mathbb{F}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Suppose that the total degree of f is $\sum_{i=1}^n d_i$ and that the coefficient in f of $\prod_{i=1}^n x_i^{d_i}$ is nonzero. Let L_i be a set of $d_i + 1$ elements of $\mathbb{F}$ for $1 \leq i \leq n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$.

Proof (continued). Now by applying Proposition 14.23 to g, we have that there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $g(\mathbf{t}) \neq 0$. Since $f(\mathbf{t} = g(\mathbf{t})$ for all such $\mathbf{t}$, then $f(\mathbf{t}) \neq 0$, as claimed. □

Corollary 14.25

Corollary 14.25. If G has an odd number of orientations D with outdegree sequence $\mathbf{d}$, then G is $(\mathbf{d} + \mathbf{1})$-list-colourable.

Proof. For D an orientation of G, we have that the sign of D is $\sigma(D) \in \{-1, +1\}$. Since we hypothesize an odd number of orientations of D with outdegree sequence $\mathbf{d}$, then the weight of $\mathbf{d}$ is $w(\mathbf{d}) = \sum \sigma(D) \neq 0$ since we can only get 0 with the same number of positive and negative orientations (and hence an even number of orientations). So in $A(G, \mathbf{x}) = \sum_{\mathbf{d}} w(\mathbf{d})\mathbf{x}^{\mathbf{d}}$ the monomial $\prod_{i=1}^n x_i^{d_i}$ has a nonzero coefficient. With lists L_i of size $d_i + 1$ for $1 \leq i \leq n$, we have by the Combinatorial Nullstellensatz (Theorem 14.24) with $f(\mathbf{x}) = A(G, \mathbf{x})$ that there is $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$. Therefore (by Note 14.6.A) there is a list colouring of G using lists $L_1, L_2, \ldots, L_n$. That is, G is $(\mathbf{d} + \mathbf{1})$-list-colourable, as claimed. □