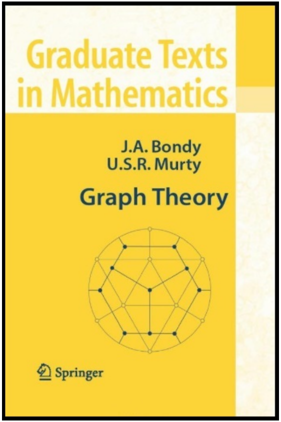
Graph Theory

Chapter 14. Vertex Colourings

14.6. The Adjacency Polynomial—Proofs of Theorems



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Proof. We give an inductive proof based on the number n of variables in f. For the base case $n = 1$, we have a polynomial in one variable and we know that such a polynomial has at most n distinct roots; see Corollary 23.5 in my online notes for Introduction to Modern Algebra (MATH 4127/5127) on Section IV.23. Factorizations of Polynomials. Since L_1 contains $d_1 + 1 = n + 1$ elements of $\mathbb{F}$, then for some $t \in L_1$ we must have $f(t) \neq 0$ so that the base case holds.

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Proposition 14.23 (continued)

Proof (continued). First, express f in the form $f = \sum_{j=0}^{d_n} f_j x_n^j$ where the coefficients f_j for $0 \leq j \leq n$ are themselves polynomials in the $n-1 = k-1$ variables $x_1, x_2, \ldots, x_{n-1}$. Since f is not the zero polynomial by hypothesis, then f_j is not the zero polynomial for some $o \leq j \leq d_n$. By the induction hypothesis, there is $t_i \in L_i$ for $1 \leq i \leq k-1$ such that $f_j(t_1, t_2, \ldots, t_{k-1}) \neq 0$. Therefore the polynomial $\sum_{j=0}^{d_k} f_j(t_1, t_2, \ldots, t_{k-1}) x_k^j$ is not the zero polynomial. This is a polynomial in one variable and so by the case $n = 1$, for some $t_k \in L_k$ we have $\sum_{j=0}^{d_k} f_j(t_1, t_2, \ldots, t_{k-1})(t_k)^j \neq 0$. That is, $f(t_1, t_2, \ldots, t_k) \neq 0$. So the induction step holds. Therefore by mathematical induction, the result holds in general. □

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Theorem 14.24. The Combinatorial Nullstellensatz

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Let f be a polynomial over a field $\mathbb{F}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Suppose that the total degree of f is $\sum_{i=1}^{n} d_i$ and that the coefficient in f of $\prod_{i=1}^{n} x_i^{d_i}$ is nonzero. Let L_i be a set of $d_i + 1$ elements of $\mathbb{F}$ for $1 \leq i \leq n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$.

Proof. For $1 \leq i \leq n$, set $f_i = \prod_{t \in L_i}(x_i - t)$ (notice that i is fixed and it is $t \in L_i$ that varies). The f_i is a polynomial of degree $|L_i| = d_i + 1$ with leading term $x_i^{d_i+1}$. Hence we can $f_i = g_i + x_i^{d_i+1}$ where g_i is a polynomial in X_i of degree at most d_i.

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Now for every monomial in f, the sum of the exponents is at most $\sum_{i=1}^{n} d_i$, so if a monomial has x_i to a power greater than or equal to $d_i + 1$ then one of the other terms of the monomial must be of the form $x_j^{e_j}$ where $e_j < d_j$. Hence, the process above of reducing the exponent on x_i to be at most d_i to create g yields a monomial of total degree strictly less than $\sum_{i=1}^{n} d_i$.

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Proof (continued). Now by applying Proposition 14.23 to g, we have that there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $g(\mathbf{t}) \neq 0$. Since $f(\mathbf{t} = g(\mathbf{t})$ for all such $\mathbf{t}$, then $f(\mathbf{t}) \neq 0$, as claimed. □

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Proof. For D an orientation of G, we have that the sign of D is $\sigma(D) \in \{-1, +1\}$. Since we hypothesize an odd number of orientations of D with outdegree sequence $\mathbf{d}$, then the weight of $\mathbf{d}$ is $w(\mathbf{d}) = \sum \sigma(D) \neq 0$ since we can only get 0 with the same number of positive and negative orientations (and hence an even number of orientations). So in $A(G, \mathbf{x}) = \sum_{\mathbf{d}} w(\mathbf{d})\mathbf{x}^{\mathbf{d}}$ the monomial $\prod_{i=1}^{n} x_i^{d_i}$ has a nonzero coefficient.

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