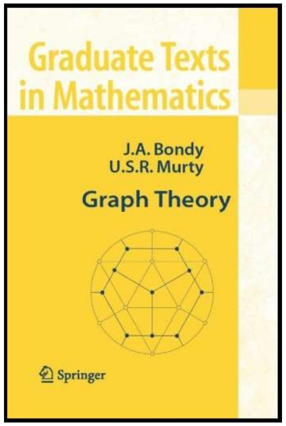
Graph Theory

Chapter 14. Vertex Colourings

14.7. The Chromatic Polynomial—Proofs of Theorems



Theorem 14.26

Theorem 14.26. For any loopless graph G, there exists a polynomial $P(G,x)$ such that $P(G,x) = C(G,k)$ for all nonnegative integers k. Moreover, if G is simple and e is any edge of G, then $P(G,x)$ satisfies the recursion formula:

$$C(G,x) = C(G \setminus e, x) - C(G/e, x). \tag{14.6}$$

The polynomial $P(G,x)$ is of degree n, with integer coefficients which alternate in sign, leading term x^n, and constant term 0.

Proof. We give an inductive proof on m, the number of edges of G. If $m = 0$ then $C(G,k) = k^n$, as described in Note 1.7.A, and the polynomial $P(G,x) = x^n$ satisfies the conditions of the theorem. This establishes the base case.

For the induction hypothesis, suppose the theorem holds for all graphs with fewer than m edges, where $m \geq 1$. Let G be a loopless graph with m edges.

Theorem 14.26 (continued 1)

Proof (continued). If G is not simple, define $P(G,x) = P(H,x)$ where H satisfies the first claim, that is there is a polynomial $P(H,x)$ such that $P(H,x) = C(H,k)$ for all nonnegative integers k, the (loopless) graph G also satisfies this condition since any proper colouring of H is also a proper colouring of G (and conversely); we do not apply the operation of edge contraction here to a graph with multiple edges (this could introduce loops). If G is simple, let e be an edge of G. Both $G \setminus e$ and G/e have $m-1$ edges (in $G \setminus e$ edge e is deleted, in G/e edge e is contracted) and are loopless. By the induction hypothesis, there exists polynomials $P(G \setminus e, x)$ and $P(G/e, x)$ such that for all nonnegative integers k, we have

$$P(G \setminus e, k) = C(G \setminus e, k) \text{ and } P(G/e, k) = c(G/e, k). \tag{14.7}$$

The induction hypothesis also gives the from of these polynomials.

Theorem 14.26 (continued 2)

Proof (continued). There are nonnegative integer coefficients $a_1, a_2, \ldots, a_{n-1}$ and $b_1, b_2, \ldots, b_{n-1}$ with

$$P(G \setminus e, x) = \sum_{i=1}^{n-1} (-1)^{n-i} a_i x^i + x^n \ \& \ P(G/e, x) = \sum_{i=1}^{n-1} (-1)^{n-i-1} b_i x^i \tag{14.8}$$

(notice that $G \setminus e$ has n vertices, but G/e has $n-1$ vertices; by the induction hypothesis we have $b_{n-1} = 1$). Define $P(G,x) = P(G \setminus e, x) - P(G/e, x)$ so that $P(G,x)$ satisfies the recursion relation (14.6). We now have

$$\begin{aligned} P(G,k) &= P(G \setminus e, k) - P(G/e, k) \text{ by (14.6) with } x = k \\ &= C(G \setminus e, k) - C(G/e, k) \text{ by (14.7)} \\ &= C(G,k) \text{ by (14.5)}. \end{aligned}$$

So $P(G,k) = C(G,k)$ for nonnegative integer k, as claimed.

Theorem 14.26 (continued 3)

Proof (continued). Now the form of $P(G,x)$ is:

$$
\begin{aligned}
P(G,x) &= P(G \setminus e, x) - P(G/e, x) \text{ by our definition of } \\
&\quad P(G,x) \text{ in this proof} \\
&= \sum_{i=1}^{n-1} (-1)^{n-i} a_i x^i + x^n - \sum_{i=1}^{n-1} (-1)^{n-i-1} b_i x^i \text{ by (14.8)} \\
&= \sum_{i=1}^{n-1} (-1)^{n-i} (a_i + b_i) x^i + x^n,
\end{aligned}
$$

so that $P(G,x)$ is of degree n, with integer coefficients which alternate in sign (notice that $a_i + b_i$ is a nonnegative integer for each i), leading term x^n, and constant term 0. This establishes the induction step. Hence, by mathematical induction, the result holds for all m and hence for all loopless graphs, or loopless simple graphs, as the hypothesis require. □