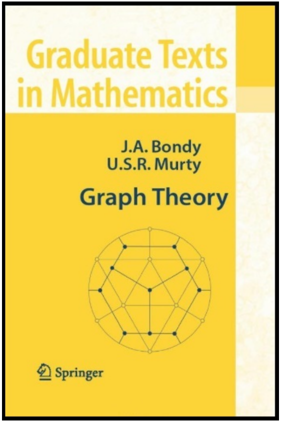
Graph Theory

Chapter 15. Colourings of Maps

15.1. Chromatic Numbers of Surfaces—Proofs of Theorems



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Proof. Let G be an arbitrary graph embedded on Σ. Without loss of generality, suppose G is colour-critical (i.e., every proper subgraph of G has a smaller chromatic number than that of G). Then by Theorem 14.7 we have $\chi \leq \delta + 1 \leq d + 1$. By Corollary 10.39 of Section 10.6. Surface Embeddings of Graphs, with m as the number of edges of G and n as the number of vertices of n, that $m \leq 3n - 3c$ where $x = x(\Sigma)$ is the Euler characteristic of Σ (see Section 10.6 again).

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$$\chi \leq 7 - 6c/n \leq |7 - 6c/n| \leq 7 + 6|c|/n \leq 7 + 6|c|.$$

So with $k = 7 + 6|c|$, we see that every graph embeddable on Σ has chromatic number at most k, as claimed. □

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For any closed surface Σ with Euler characteristic $c \leq 0$ we have

$$\chi(\Sigma) \leq \frac{1}{2}\left(7 + \sqrt{49 - 24c}\right).$$

Proof. In the proof of Theorem 15.1.A, we saw for a graph with n vertices embedded on closed surface Σ that $\chi(\Sigma) \leq 7 - 6c/n$ where $c = c(\Sigma)$ is the Euler characteristic of Σ. Since $\chi \leq n$ then $\chi \leq 7 - 6c/n$ implies $\chi \leq 7 - 6c/n \leq 7 - 6c/\chi$, or $\chi^2 - 7\chi + 6c \leq 0$. The graph of polynomial $p(\chi) = \chi^2 - 7\chi + 6c$ is concave up and has roots $\dfrac{1}{2}\left(7 \pm \sqrt{49 - 24c}\right)$. Now $\dfrac{1}{2}\left(7 \pm \sqrt{49 - 24c}\right) \leq 0$, so inequality $\chi^2 - 7\chi + 6c \leq 0$ implies that $\chi \leq \frac{1}{2}\left(7 + \sqrt{49 - 24c}\right)$, as claimed. □

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