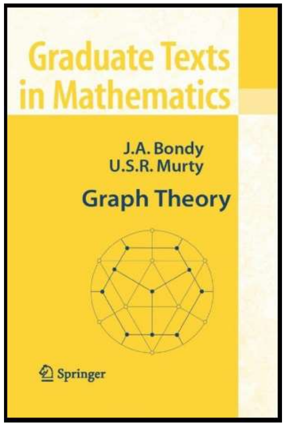
Graph Theory

Chapter 15. Colourings of Maps

15.2. The Four-Colour Theorem—Proofs of Theorems



Theorem 15.2

Proposition 15.2. Let G be a smallest counterexample to the Four-Colour Theorem. Then

(i) G is 5-critical,

(ii) G is a triangulation, and

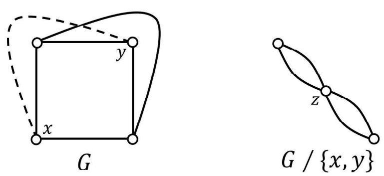
(iii) G has no vertex of degree less than four.

Proof. (i) By the definition of 5-critical, if G is not 5-critical then it has a proper subgraph that is 5-critical, contradicting the minimality of $v(G) + e(G)$ given in Note 15.2.A(ii). So G must be 5-critical.

(ii) ASSUME G is not a triangulation. Then it has a face whose boundary is a cycle C of length greater than three. Since G is planar, at least two vertices of C, say x and y, are nonadjacent in G (see the figure below).

Theorem 15.2 (continued 1)

Proof (continued).



The graph $G/\{x, y\}$ obtained by identifying x and y into a single vertex z is a planar graph with fewer vertices than G, and the same number of edges. Since G is 5-critical by (i), then $G/\{x, y\}$ is 4-colourable with, say, colouring c. Now G can be 4-coloured by assigning colour $c(v)$ to each $v \in V(G) \setminus \{x, y\}$ and assigning colour $c(z)$ to vertices x and y. This is a CONTRADICTION to the (assumed) fact that G is a counterexample to the Four-Colour Theorem. So the assumption that G is not a triangulation is false, and hence G is a triangulation, as claimed.

Theorem 15.2 (continued 2)

Proposition 15.2. Let G be a smallest counterexample to the Four-Colour Theorem. Then

(i) G is 5-critical,

(ii) G is a triangulation, and

(iii) G has no vertex of degree less than four.

Proof (continued). (iii) Since G is 5-critical by (i), then Theorem 14.7 implies $\delta \geq k - 1 = 5 - 1 = 4$, as claimed. □

Theorem 15.3

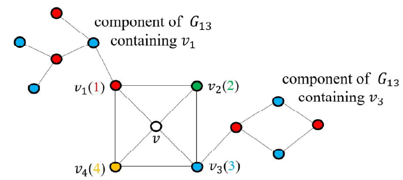
Theorem 15.3. A smallest counterexample G to the Four-Colour Theorem has no vertex of degree four.

Proof. ASSUME G has a vertex v of degree four. Then $G - v$ is a proper subgraph of G and, since G is 5-critical by Proposition 15.2(i), then $G - v$ is 4-colourable. Let the colour classes of a 4-colouring of $G - v$ be (V_1, V_2, V_3, V_4). Because G itself is not 4-colourable, then v must be adjacent to one vertex of each colour. Without loss of generality, we may assume that the neighbors of v in clockwise order (so we can draw a picture) are v_1, v_2, v_3, v_4 where $v_i \in V_i$ for $1 \le i \le 4$.

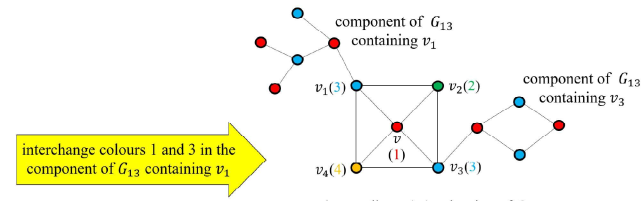
Denote by G_{ij} the subgraph of G induced by the set of vertices $V_i \cup V_j$ (so every vertex of G_{ij} is either colour i or colour j). We claim that v_i and v_j are in the same connected component of G_{ij}. If not, consider the component of G_{ij} that contains v_i. By interchanging colours i and j in this component, we obtain a new 4-colouring of $G - v$ in which only three colours (all but colour i) are assigned to the neighbors of v. See the figure below.

Theorem 15.3 (continued 1)

Proof (continued). But then we could assign colour i to vertex v giving a 4-colouring of G, contradicting the (assumed) fact that it is not 4-colourable. So our claim that v_i and v_j are in the same component of G_{ij} holds.



v_1 and v_3 in different components of G_{13}



a (contradictory) 4-colouring of G

Theorem 15.3 (continued 2)

Proof (continued). Let P_{ij} be a $v_i v_j$-path in G_{ij} and let C denote the cycle $vv_1P_{13}v_3v$ (see Figure 15.5).

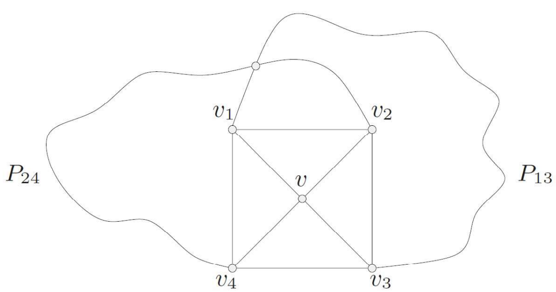


Fig. 15.5. Kempe's proof of the case $d(v) = 4$

Because C separates v_2 and v_4 (in the Figure 15.5 we have $v_2 \in \text{int}(C)$ and $v_4 \in \text{ext}(C)$), then by the Jordan Curve Theorem (Theorem 10.1), path P_{24} meets C in some point.

Theorem 15.3 (continued 3)

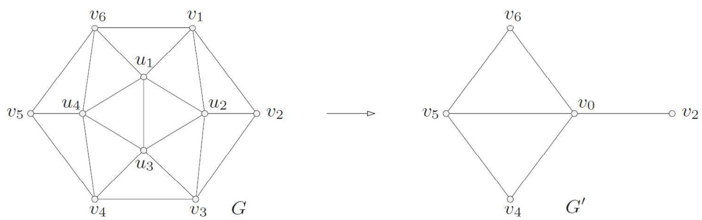
Theorem 15.3. A smallest counterexample G to the Four-Colour Theorem has no vertex of degree four.

Proof (continued). Because G is a plane graph by hypothesis, this point must be a vertex of G. But the vertices of path P_{13} are all either colour 1 or 3 and vertices of path P_{24} are all either colour 2 or 4. The existence of a vertex on both paths is therefore a CONTRADICTION. So the assumption that G has a vertex of degree four is false, and hence G has no vertex of degree four, as claimed. □

Theorem 15.7

Theorem 15.7. The Birkhoff diamond is reducible.

Proof. ASSUME G is a smallest conterexample to the Four-Colour Theorem with the Birkhoff diamond as a configuration. Because G is essentially 6-connected by Theorem 15.6 then, by Exercise 15.2.3, no edge of G can join nonconsecutive vertices on the boundary cycle of the Birkhoff diamond. Consider the plane graph G' derived from G by deleting the four internal bridge vertices (vertices u_1, u_2, u_3, u_4 in Figure 15.7), identifying vertices v_1 and v_3 to form a new vertex v_0, deleting one of the two multiple edges between v_0 and v_2, and joining v_0 and v_5; see Figure 15.8:



Theorem 15.7 (continued 1)

Proof (continued). Since no edge of G can join nonconsecutive vertices on the bounding cycle (in particular, no edge of G bounds v_1 and v_3) then G' contains no loops.

Because $v(G') + e(G') < v(G) + e(G)$ and G is a smallest counterexample to the Four-Colour Theorem, there exists a 4-colouring c' of G'. The colouring c' gives rise to a partial 4-colouring of G (in fact, a 4-colouring of $G - \{u_1, u_2, u_3, u_4\}$ since v_1 and v_2 are not adjacent in G) in which:

(1) v_1 and v_3 receive the same colour, say 1,

(2) v_5 and receives a colour different from 1, say 2,

(3) v_3 receives a colour different fom 1, without loss of generality, either 2 or 3 (that is, either the same colour as v_5 or a different colour from the colour of v_5 which we take without loss of generality to be 3; we could also choose v_5 to be colour 4),

(4) v_4 and v_6 each receives a colour different from 1 or 2, namely either 3 or 4.

Theorem 15.7 (continued 2)

Proof (continued). We expect $3 \times 2 \times 2 = 12$ different colourings of the bounding cycle $C = v_1 v_2 v_3 v_4 v_5 v_6 v_1$. We can interchange colours 3 and 4, reducing the number of colourings to five:

	v_1	v_2	v_3	v_4	v_5	v_6
c_1	1	2	1	3	2	3
c_2	1	2	1	4	2	3
c_3	1	3	1	4	2	3
c_4	1	3	1	4	2	4
c_5	1	3	1	3	2	3

Interchanging colours 3 and 4 on vertices v_4 and v_6 gives new colourings from c_1, c_2, and c_3 (for three more colourings). Replacing colour 3 with colour 4 on vertex v_2 gives new colourings from c_3, c_4, c_5 (for three more colourings); then also interchanging colours 3 and 4 on vertices v_4 and v_6 in the modified colouring of c_4 gives a new colouring (for a total of $5 + 3 + 3 + 1 = 12$ colourings, as expected).

Theorem 15.7 (continued 3)

Proof (continued). In colourings c_1 through c_4 it is straightforward to show that the colouring of $G - \{u_1, u_2, u_3, u_4\}$ can be extended to a colouring of G, as is to be shown in Exercise 15.2.4(a). Consider now the colouring c_5. In this case we will use a Kempe interchange to modify c_5 to create a 4-colouring of $G - \{u_1, u_2, u_3, u_4\}$ that can be extended to a 4-colouring of G.

First, consider the bipartite graph G_{34} induced by the vertices coloured 3 or 4. We claim that v_2, v_4, and v_6 (each of colour 3) belong to the same connected component H of G_{34}. Suppose v_2 is in some component of G_{34}, but neither v_4 nor v_6 are in this component. By swapping the colours 3 and 4 in this component, we obtain a colouring of "type" c_4 (we need to then use symmetry and interchange colours 3 and 4 to get colouring c_4; thus the "type" term). The other cases (a component of G_{34} containing v_4 but neither v_5 nor v_6, and a component of G_{34} containing v_6 but neither v_2 nor v_4) are addressed in Exercise 15.2.4(b). Therefore we can assume that v_2, v_4, and v_6 belong to the same component H of G_{34}, as claimed.

Theorem 15.7 (continued 4)

Theorem 15.7. The Birkhoff diamond is reducible.

Proof (continued). Second, we have that H is an outer bridge of C in G with vertices of attachment v_2, v_4, and v_6 (by definition, a bridge is a connected graph so that's why we are concerned with a component of G_{34}; notice that H cannot be an inner bridge, as seen in Figure 15.7). Next, consider the bipartite subgraph G_{12} of G induced by the vertices of colours 1 and 2. If there were a component of G_{12} which contained both v_3 and v_5, then this component would be an outer bridge of C overlapping H, which cannot happen (by the Jordan Curve Theorem, Theorem 10.1; see Figure 15.7). So the component H' of G_{12} which contains v_3 does not contain v_5. Interchanging colours 1 and 2 in H', we obtain a new partial 4-colouring of G.

Theorem 15.7 (continued 5)

Theorem 15.7. The Birkhoff diamond is reducible.

Proof (continued). In this colouring v_1 has colour 1, v_3 and v_5 have colour 2, and vertices v_2, v_4, v_6 are colour 3 (we have not changed the original colours of vertices v_1, v_2, v_4, v_5, v_6, but we have changed v_3 from colour 1 to colour 2). This partial colouring of G can be extended to a 4-colouring of G by assigning colour 2 to u_1, colour 4 to u_2 and u_4, and colour 1 to u_3. But G is a smallest counterexample to the Four-Colour Theorem and so G is not 4-colourable, a CONTRADICTION. So the assumption that a smallest counterexample to the Four-Colour Theorem has the Birkhoff diamond as a configuration is false. That is (by definition), the Birkhoff diamond is reducible, as claimed. □

Theorem 15.2.A

Theorem 15.2.A. A planar graph is 3-colourable if it contains no cycles of length k for $4 \leq k \leq 11$.

Proof. ASSUME the claim is false. Let G be a smallest counterexample (that is, the sum $v(G) + e(G)$ is as small as possible among all counterexamples) to eh assertion. Since G is a smallest counterexample, it does not have a cut vertex (or else we could consider the two subgraphs of G which are joined at the cut vertex and delete the vertices in the component with the lesser [or equal] chromatic number from G, except fo the cut vertex,; the resulting graph is smaller than G and yet has the same chromatic number as G, contradicting the minimality of G). That is, G is 2-connected. If $\delta(G) = 2$ then G is a cycle and so is 3-colourable, contradicting the fact that G is a counterexample. Therefore $\delta(G) \geq 3$. We assign charges to both vertices and faces based on their degrees. For $v \in V$ assign the charge $d(v) - 6$ and for face $f \in F$ (i.e., f is a face in a planar embedding of G) assign the charge $2d(v) - 6$.

Theorem 15.2.A (continued)

Theorem 15.2.A. A planar graph is 3-colourable if it contains no cycles of length k for $4 \leq k \leq 11$.

Proof. In Exercise 15.2.A it is to be verified that the total charge assigned to vertices and faces is -12. For the discharging algorithm, each face of degree twelve or more transfers a charge of 3/2 to each of the vertices incident to the face. Since G is 2-connected, by Theorem 10.7 all faces of G are bounded by cycles. Because G has no 4-cycles, no edge of G can be incident with two triangles. Thus each vertex v is incident with at least $\lceil v/2 \rceil$ distinct faces of degree twelve or more (adn at most $\lfloor d(v)/2 \rfloor$ triangles). In Exercise 15.2.A it is to be shown that after the transfer of charges, all vertices and faces have nonnegative charges. Set $\mathcal{U}$ of unavoidable configurations is then empty. But the smallest counterexample must contain at least one element of $\mathcal{U}$, a CONTRADICTION. So the assumption that a (smallest) counterexample exists is false, and the claim holds. □