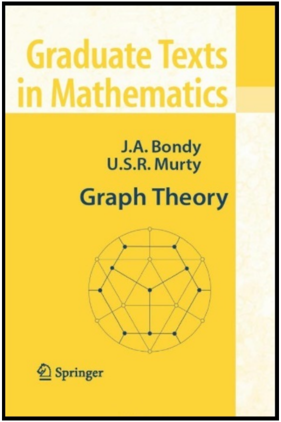
Graph Theory

Chapter 15. Colourings of Maps

15.2. The Four-Colour Theorem—Proofs of Theorems



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Theorem 15.2

Proposition 15.2. Let G be a smallest counterexample to the Four-Colour Theorem. Then

(i) G is 5-critical,

(ii) G is a triangulation, and

(iii) G has no vertex of degree less than four.

Proof. (i) By the definition of 5-critical, if G is not 5-critical then it has a proper subgraph that is 5-critical, contradicting the minimality of $v(G) + e(G)$ given in Note 15.2.A(ii). So G must be 5-critical.

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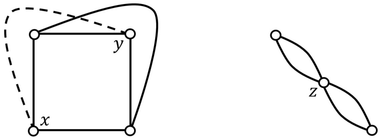
Theorem 15.2 (continued 1)

Proof (continued).



G $\qquad$ $G/\{x,y\}$

The graph $G/\{x,y\}$ obtained by identifying x and y into a single vertex z is a planar graph with fewer vertices than G, and the same number of edges. Since G is 5-critical by (i), then $G/\{x,y\}$ is 4-colourable with, say, colouring c. Now G can be 4-coloured by assigning colour $c(v)$ to each $v \in V(G) \setminus \{x,y\}$ and assigning colour $c(z)$ to vertices x and y. This is a CONTRADICTION to the (assumed) fact that G is a counterexample to the Four-Colour Theorem. So the assumption that G is not a triangulation is false, and hence G is a triangulation, as claimed.

Theorem 15.2 (continued 2)

Proposition 15.2. Let G be a smallest counterexample to the Four-Colour Theorem. Then

(i) G is 5-critical,

(ii) G is a triangulation, and

(iii) G has no vertex of degree less than four.

Proof (continued). (iii) Since G is 5-critical by (i), then Theorem 14.7 implies $\delta \geq k - 1 = 5 - 1 = 4$, as claimed. $\square$

Theorem 15.3

Theorem 15.3. A smallest counterexample G to the Four-Colour Theorem has no vertex of degree four.

Proof. ASSUME G has a vertex v of degree four. Then $G - v$ is a proper subgraph of G and, since G is 5-critical by Proposition 15.2(i), then $G - v$ is 4-colourable. Let the colour classes of a 4-colouring of $G - v$ be (V_1, V_2, V_3, V_4). Because G itself is not 4-colourable, then v must be adjacent to one vertex of each colour. Without loss of generality, we may assume that the neighbors of v in clockwise order (so we can draw a picture) are v_1, v_2, v_3, v_4 where $v_i \in V_i$ for $1 \leq i \leq 4$.

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Denote by G_{ij} the subgraph of G induced by the set of vertices $V_i \cup V_j$ (so every vertex of G_{ij} is either colour i or colour j). We claim that v_i and v_j are in the same connected component of G_{ij}. If not, consider the component of G_{ij} that contains v_i. By interchanging colours i and j in this component, we obtain a new 4-colouring of $G - v$ in which only three colours (all but colour i) are assigned to the neighbors of v. See the figure below.

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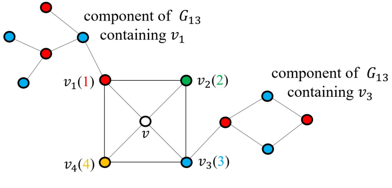
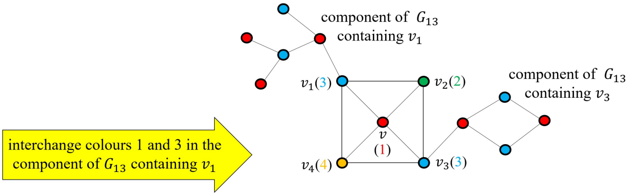
Theorem 15.3 (continued 1)

Proof (continued). But then we could assign colour i to vertex v giving a 4-colouring of G, contradicting the (assumed) fact that it is not 4-colourable. So our claim that v_i and v_i are in the same component of G_{ij} holds.



v_1 and v_3 in different components of G_{13}



a (contradictory) 4-colouring of G

Theorem 15.3 (continued 2)

Proof (continued). Let P_{ij} be a $v_i v_j$-path in G_{ij} and let C denote the cycle $vv_1 P_{13} v_3 v$ (see Figure 15.5).

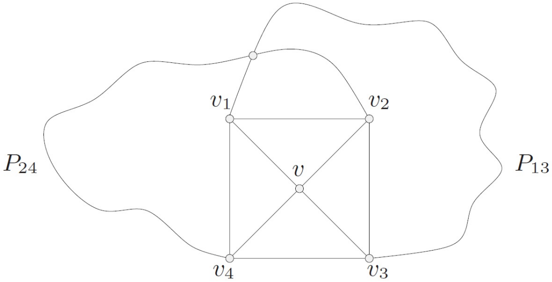


Fig. 15.5. Kempe's proof of the case $d(v) = 4$

Because C separates v_2 and v_4 (in the Figure 15.5 we have $v_2 \in \text{int}(C)$ and $v_4 \in \text{ext}(C)$), then by the Jordan Curve Theorem (Theorem 10.1), path P_{24} meets C in some point.

Theorem 15.3 (continued 3)

Theorem 15.3. A smallest counterexample G to the Four-Colour Theorem has no vertex of degree four.

Proof (continued). Because G is a plane graph by hypothesis, this point must be a vertex of G. But the vertices of path P_{13} are all either colour 1 or 3 and vertices of path P_{24} are all either colour 2 or 4. The existence of a vertex on both paths is therefore a CONTRADICTION. So the assumption that G has a vertex of degree four is false, and hence G has no vertex of degree four, as claimed. $\square$

Theorem 15.7

Theorem 15.7. The Birkhoff diamond is reducible.

Proof. ASSUME G is a smallest conterexample to the Four-Colour Theorem with the Birkhoff diamond as a configuration. Because G is essentially 6-connected by Theorem 15.6 then, by Exercise 15.2.3, no edge of G can join nonconsecutive vertices on the boundary cycle of the Birkhoff diamond. Consider the plane graph G' derived from G by deleting the four internal bridge vertices (vertices u_1, u_2, u_3, u_4 in Figure 15.7), identifying vertices v_1 and v_3 to form a new vertex v_0, deleting one of the two multiple edges between v_0 and v_2, and joining v_0 and v_5; see Figure 15.8:

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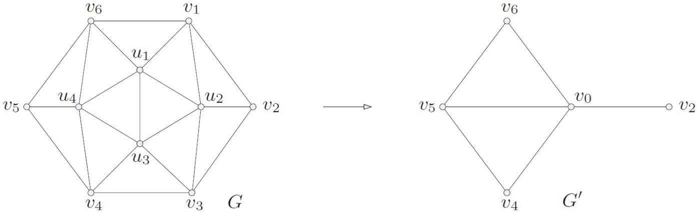

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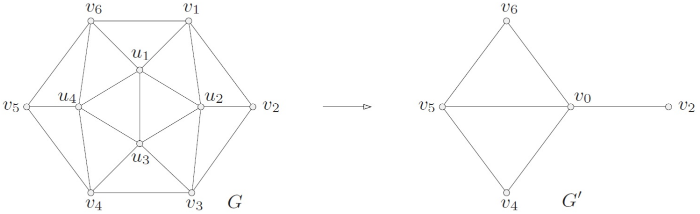

Theorem 15.7 (continued 1)

Proof (continued). Since no edge of G can join nonconsecutive vertices on the bounding cycle (in particular, no edge of G bounds v_1 and v_3) then G' contains no loops.

Because $v(G') + e(G') < v(G) + e(G)$ and G is a smallest counterexample to the Four-Colour Theorem, there exists a 4-colouring c' of G'. The colouring c' gives rise to a partial 4-colouring of G (in fact, a 4-colouring of $G - \{u_1, u_2, u_3, u_4\}$ since v_1 and v_2 are not adjacent in G) in which:

(1) v_1 and v_3 receive the same colour, say 1,

(2) v_5 and receives a colour different from 1, say 2,

(3) v_3 receives a colour different fom 1, without loss of generality, either 2 or 3 (that is, either the same colour as v_5 or a different colour from the colour of v_5 which we take without loss of generality to be 3; we could also choose v_5 to be colour 4),

(4) v_4 and v_6 each receives a colour different from 1 or 2, namely either 3 or 4.

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Theorem 15.7 (continued 2)

Proof (continued). We expect $3 \times 2 \times 2 = 12$ different colourings of the bounding cycle $C = v_1 v_2 v_3 v_4 v_5 v_6 v_1$. We can interchange colours 3 and 4, reducing the number of colourings to five:

	v_1	v_2	v_3	v_4	v_5	v_6
c_1	1	2	1	3	2	3
c_2	1	2	1	4	2	3
c_3	1	3	1	4	2	3
c_4	1	3	1	4	2	4
c_5	1	3	1	3	2	3

Interchanging colours 3 and 4 on vertices v_4 and v_6 gives new colourings from c_1, c_2, and c_3 (for three more colourings). Replacing colour 3 with colour 4 on vertex v_2 gives new colourings from c_3, c_4, c_5 (for three more colourings); then also interchanging colours 3 and 4 on vertices v_4 and v_6 in the modified colouring of c_4 gives a new colouring (for a total of $5 + 3 + 3 + 1 = 12$ colourings, as expected).

Theorem 15.7 (continued 3)

Proof (continued). In colourings c_1 through c_4 it is straightforward to show that the colouring of $G - \{u_1, u_2, u_3, u_4\}$ can be extended to a colouring of G, as is to be shown in Exercise 15.2.4(a). Consider now the colouring c_5. In this case we will use a Kempe interchange to modify c_5 to create a 4-colouring of $G - \{u_1, u_2, u_3, u_4\}$ that can be extended to a 4-colouring of G.

First, consider the bipartite graph G_{34} induced by the vertices coloured 3 or 4. We claim that v_2, v_4, and v_6 (each of colour 3) belong to the same connected component H of G_{34}. Suppose v_2 is in some component of G_{34}, but neither v_4 nor v_6 are in this component. By swapping the colours 3 and 4 in this component, we obtain a colouring of "type" c_4 (we need to then use symmetry and interchange colours 3 and 4 to get colouring c_4; thus the "type" term). The other cases (a component of G_{34} containing v_4 but neither v_5 nor v_6, and a component of G_{34} containing v_6 but neither v_2 nor v_4) are addressed in Exercise 15.2.4(b). Therefore we can assume that v_2, v_4, and v_6 belong to the same component H of G_{34}, as claimed.

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Theorem 15.7 (continued 4)

Theorem 15.7. The Birkhoff diamond is reducible.

Proof (continued). Second, we have that H is an outer bridge of C in G with vertices of attachment v_2, v_4, and v_6 (by definition, a bridge is a connected graph so that's why we are concerned with a component of G_{34}; notice that H cannot be an inner bridge, as seen in Figure 15.7). Next, consider the bipartite subgraph G_{12} of G induced by the vertices of colours 1 and 2. If there were a component of G_{12} which contained both v_3 and v_5, then this component would be an outer bridge of C overlapping H, which cannot happen (by the Jordan Curve Theorem, Theorem 10.1; see Figure 15.7). So the component H' of G_{12} which contains v_3 does not contain v_5. Interchanging colours 1 and 2 in H', we obtain a new partial 4-colouring of G.

Theorem 15.7 (continued 4)

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Theorem 15.7 (continued 5)

Theorem 15.7. The Birkhoff diamond is reducible.

Proof (continued). In this colouring v_1 has colour 1, v_3 and v_5 have colour 2, and vertices v_2, v_4, v_6 are colour 3 (we have not changed the original colours of vertices v_1, v_2, v_4, v_5, v_6, but we have changed v_3 from colour 1 to colour 2). This partial colouring of G can be extended to a 4-colouring of G by assigning colour 2 to u_1, colour 4 to u_2 and u_4, and colour 1 to u_3. But G is a smallest counterexample to the Four-Colour Theorem and so G is not 4-colourable, a CONTRADICTION. So the assumption that a smallest counterexample to the Four-Colour Theorem has the Birkhoff diamond as a configuration is false. That is (by definition), the Birkhoff diamond is reducible, as claimed. $\square$

Theorem 15.2.A

Theorem 15.2.A. A planar graph is 3-colourable if it contains no cycles of length k for $4 \le k \le 11$.

Proof. ASSUME the claim is false. Let G be a smallest counterexample (that is, the sum $v(G) + e(G)$ is as small as possible among all counterexamples) to eh assertion. Since G is a smallest counterexample, it does not have a cut vertex (or else we could consider the two subgraphs of G which are joined at the cut vertex and delete the vertices in the component with the lesser [or equal] chromatic number from G, except fo the cut vertex,; the resulting graph is smaller than G and yet has the same chromatic number as G, contradicting the minimality of G). That is, G is 2-connected.

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Theorem 15.2.A (continued)

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Proof. In Exercise 15.2.A it is to be verified that the total charge assigned to vertices and faces is -12. For the discharging algorithm, each face of degree twelve or more transfers a charge of $3/2$ to each of the vertices incident to the face. Since G is 2-connected, by Theorem 10.7 all faces of G are bounded by cycles. Because G has no 4-cycles, no edge of G can be incident with two triangles. Thus each vertex v is incident with at least $\lceil v/2 \rceil$ distinct faces of degree twelve or more (adn at most $\lfloor d(v)/2 \rfloor$ triangles). In Exercise 15.2.A it is to be shown that after the transfer of charges, all vertices and faces have nonnegative charges. Set $\mathcal{U}$ of unavoidable configurations is then empty. But the smallest counterexample must contain at least one element of $\mathcal{U}$, a CONTRADICTION. So the assumption that a (smallest) counterexample exists is false, and the claim holds. □

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