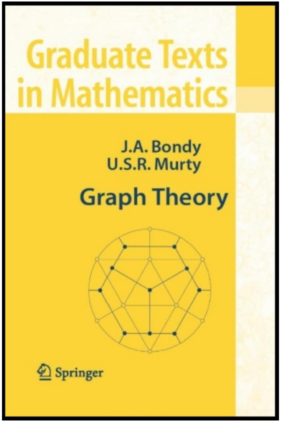
Graph Theory

Chapter 15. Colourings of Maps

15.3. List Colourings of Planar Graphs—Proofs of Theorems



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(i) $|L(x)| = |L(y)| = 1$, where $L(x) \neq L(y)$,
(ii) $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{x, y\}$, and
(iii) $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$.

Then G is L-colourable.

Proof. We give an inductive proof on the number $v(G)$ of vertices of G (notice that because of the triangulation condition, $v(G) \geq 3$). For the base case $v(G) = 3$, we have $G = C$ and the result holds trivially. For the induction hypothesis, suppose the result holds for all graphs G satisfying the hypotheses such that $3 \leq v(G) \leq \ell$. Suppose G is a graph satisfying the hypotheses where $v(G) = \ell + 1$ (so $v(G) > 3$). Let z and x' be the immediate predecessors of x on C (see Figure 15.12(a) below).

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Theorem 15.3.8 (continued 1)

Proof (continued).
Consider first the case where x' has a neighbor y' on C other than x and z (Figure 15.12(a)). In this case $C_1 = x'Cy'x'$ and $C_2 = x'y'Cx'$ are two cycles of G, and G is the union of the near-triangulation G_1 consisting of C_1 together with its interior, and the near-triangulation G_2 consisting of C_2 together with its interior. Now let L_2 be the functions on $V(G_2)$ defined by $L_2(x') = \{c_1(x')\}$, $L_2(y') = \{c_1(y')\}$, and $L_2(v) = L(v)$ for $v \in V(G_2) \setminus \{x', y'\}$. Then the hypotheses of the theorem are satisfied by G_2 and L_2 (where x' and y' of G_2 play the roles of x and y, respectively, in the hypotheses).

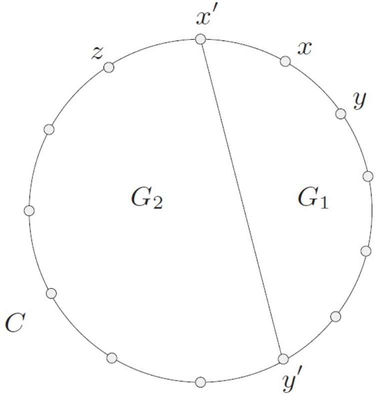

Fig. 15.12.(a)

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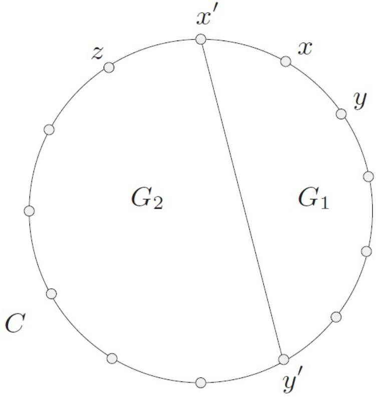

Fig. 15.12.(a)

Then the hypotheses of the theorem are satisfied by G_2 and L_2 (where x' and y' of G_2 play the roles of x and y, respectively, in the hypotheses).

Theorem 15.3.8 (continued 2)

Proof (continued). Since $v(G_2) \leq \ell$, by the induction hypothesis there is an L_2-colouring c_2 of G_2. By the definition of L_2, the colourings c_1 and c_2 assign the same colours to x' and y' (the only two vertices common to G_1 and G_2; we are establishing that the colouring we define on G is, in the terminology of modern algebra, well-defined). Thus the function c defined by $c(v) = c_1(v)$ for $v \in V(G_1)$ and $c(v) = c_2(v)$ for $v \in V(G_2) \setminus V(G_1)$ is an L-colouring of G. So the induction step is verified in this first case.

Consider the second case that the neighbors of x' other than x and z lie on a path xPz internally disjoint from C (such a path exists because of the triangulation property of G). See Figure 15.12(b) below. In this case, $G' = G - x'$ is a near-triangulation whose outer face is bounded by the cycle $C' = xCz\overleftarrow{P}x$ (we need to reverse the xPz path for this).

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Consider the second case that the neighbors of x' other than x and z lie on a path xPz internally disjoint from C (such a path exists because of the triangulation property of G). See Figure 15.12(b) below. In this case, $G' = G - x'$ is a near-triangulation whose outer face is bounded by the cycle $C' = xCz\overleftarrow{P}x$ (we need to reverse the xPz path for this).

Theorem 15.3.8 (continued 3)

Proof (continued).
In this case, $G' = G - x'$ is a near-triangulation whose outer face is bounded by the cycle $C' = xCz\overleftarrow{P}x$ (we need to reverse the xPz path for this). By hypothesis, $|L(x')| \geq 3$ and $|L(x)| = 1$, so $|L(x') \setminus L(x)| \geq 2$. Let α and β be two distinct colours in $L(x') \setminus L(x)$.

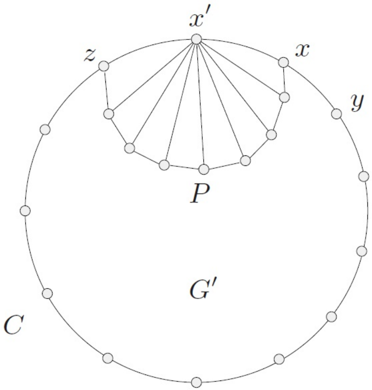

Fig. 15.12.(b)

Define L' on $V(G')$ as $L'(v) = L(v) \setminus \{\alpha, \beta\}$ for $v \in V(P) \setminus \{x, z\}$ and $L'(v) = L(v)$ for all other vertices of G'. Then L' satisfies the hypotheses of the theorem (notice that for $v \in V(P) \setminus \{x, z\}$ we have $|L'(v)| = |L(v) \setminus \{\alpha, \beta\}| \geq |L(v)| - |\{\alpha, \beta\}|$ $\geq 5 - 2 = 3$, as needed since such v are on cycle C'). By the induction hypothesis (since $v(G') = \ell$), there is an L'-colouring c' of G'.

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In this case, $G' = G - x'$ is a near-triangulation whose outer face is bounded by the cycle $C' = xCz\overleftarrow{P}x$ (we need to reverse the xPz path for this). By hypothesis, $|L(x')| \geq 3$ and $|L(x)| = 1$, so $|L(x') \setminus L(x)| \geq 2$. Let α and β be two distinct colours in $L(x') \setminus L(x)$. Define L' on $V(G')$ as $L'(v) = L(v) \setminus \{\alpha, \beta\}$ for $v \in V(P) \setminus \{x, z\}$ and $L'(v) = L(v)$ for all other vertices of G'. Then L' satisfies the hypotheses of the theorem (notice that for

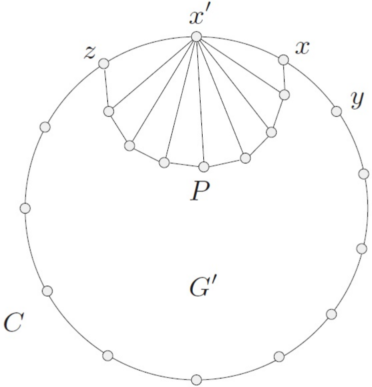

Fig. 15.12.(b)

$v \in V(P) \setminus \{x, z\}$ we have $|L'(v)| = |L(v) \setminus \{\alpha, \beta\}| \geq |L(v)| - |\{\alpha, \beta\}|$ $\geq 5 - 2 = 3$, as needed since such v are on cycle C'). By the induction hypothesis (since $v(G') = \ell$), there is an L'-colouring c' of G'.

Theorem 15.3.8 (continued 4)

Theorem 15.3.8. Let G be a near-triangulation whose outer face is bounded by a cycle of C, and let x and y be consecutive vertices of C. Suppose that $L : V \to 2^{\mathbb{N}}$ (where $2^{\mathbb{N}}$ denotes the power set of $\mathbb{N}$) is an assignment of lists of colours to the vertices of G such that:

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(iii) $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$.

Then G is L-colourable.

Proof (continued). One of the colours α and β is different from $c'(z)$ (and by the choice no other neighbors of x' in G is colour α or β), so we can assign that colour that colour to x'. This extension of c' is an L-colouring c of G. So the induction step is verified in this second case. So, by mathematical induction, the result holds for all graphs satisfying the hypotheses. □