Graph Theory

Chapter 15. Colourings of Maps

15.4. Hadwiger's Conjecture—Proofs of Theorems

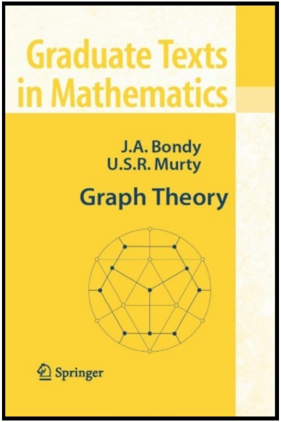

Table of contents

1. Theorem 15.12
2. Theorem 15.13
3. Corollary 15.14

Theorem 15.12

Theorem 15.12. Every 4-chromatic graph contains a K_4-subdivision.

Proof. Let G be a 4-chromatic graph. Then G contains a 4-critical subgraph F. By Theorem 14.7, $\delta(F) \geq 3$. By Exercise 10.1.5, F contains a subdivision of K_4. This subdivision of K_4 is also a subgraph of G, as claimed. □

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Theorem 15.13. Every simple graph G with $m \geq 2^{k-3}n$ (that is, $e(G) \geq 2^{k-2}v(G)$) has a K_k-minor.

Proof. We give an inductive proof on the number of edges m of G. If $k = 1$ then G trivially has a K_1-minor. If $k = 2$ and $m \geq n/2$ (where $n \geq 2$, since G is simple) then G trivially has a K_2-minor. When $k = 3$ and $m \geq n$ we have by Theorem 4.3 then G is not a tree and so it contains a cycle. By a sequence of vertex deletions and edge contradictions, we have that G has a K_2-minor. This we may assume $k \geq 4$.

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Theorem 15.13 (continued 1)

Theorem 15.13. Every simple graph G with $m \geq 2^{k-3}n$ (that is, $e(G) \geq 2^{k-2}v(G)$) has a K_k-minor.

Proof (continued). For the induction hypotheses, suppose the result holds for all $m \leq \ell$ (where $\ell = 10$). Let G be a graph with n vertices and $m = \ell + 1$ edges where $m \geq 2^{k-3}n$. We consider two cases.

First, suppose G has an edge e which lies in at most $2^{k-3} - 1$ triangles. Then the underlying simple graph of G/e (notice that G/e itself might have multiple edges) has $n - 1$ vertices. Each triangle of G which contains edge e results in a double edge in G/e, which is replaced by a single edge in the underlying simple graph of G/e. So this underlying simple graph has at least $m - 2^{k-3} \geq 2^{k-3}n - 2^{k-3} = 2^{k-3}(n-1)$ edges. The underlying simple graph of G/e has less edges than graph G, so by the induction hypothesis it has a K_k-minor. This K_k-minor of G/e is also a minor of G and the induction step holds in this case.

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Theorem 15.13 (continued 2)

Proof (continued). Second suppose that each edge of G lies in at least 2^{k-3} triangles. For $e \in E$, denote by $t(e)$ the number of triangles containing e. Any edge e of G is in the subgraph $G[N(v)]$ induced by the neighbors of a vertex v if and only if v is the 'apex' of a triangle whose 'base' is e (i.e., v is a vertex of some triangle containing e, other than the two ends of edge e). Notice that edge e could be in several subgraphs $G[N(v)]$; in fact, it is such a subgraph $t(e)$ times (i.e., for $t(e)$ different vertices v). Hence we have

$$\begin{aligned}\sum_{v\in V}|E(G[N(v)])| &= \sum_{e\in E} t(e) = 2^{k-3}m \text{ since each of the } m \text{ edges} \\ &\quad \text{lies in at least } 2^{k-3} \text{ triangles in this case} \\ &= 2^{k-3}\sum_{v\in V} d(v)/2 \text{ by Theorem 1.1} \\ &= \sum_{v\in V} 2^{k-4}d(v).\end{aligned}$$

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Theorem 15.13 (continued 3)

Theorem 15.13. Every simple graph G with $m \geq 2^{k-3}n$ (that is, $e(G) \geq 2^{k-2}v(G)$) has a K_k-minor.

Proof (continued). This inequality implies that G has at least one vertex v such that its neighborhood subgraph $H = G[N(v)]$ satisfies

$$e(H) \geq 2^{k-4}d(v) = 2^{k-4}v(H). \qquad (*)$$

Since H is a subgraph of G which excludes vertex v and all edges incident to v, then H has less edges than G. So, by the induction hypothesis (and $(*)$), H has a K_{k-1}-minor. This K_{k-1} along with vertex v and all edges between v and vertices of K_{k-1} (i.e., $K_{k-1} \vee K_1$ where $V(K_1) = \{v\}$) gives a K_k-minor of G. In both cases, G has a K_k-minor and so by mathematical induction the claim holds for all graphs satisfying the hypotheses. □

Corollary 15.14

Corollary 15.14. For $k \geq 2$, every $(2^{k-2}+1)$-chromatic graph has a K_k-minor.

Proof. Let G be a $(2^{k-2}+1)$-chromatic graph and let F be a $(2^{k-2}+1)$-critical subgraph of G. By Theorem 14.7, $\delta(F) \geq (2^{k-2}+1) - 1 = 2^{k-2}$, and so $e(F) \geq \delta(F)v(F)/2 \geq 2^{2k-3}v(F)$. Hence by Theorem 15.7, F has a K_k-minor. Therefore, G has a K_k-minor, as claimed. □

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