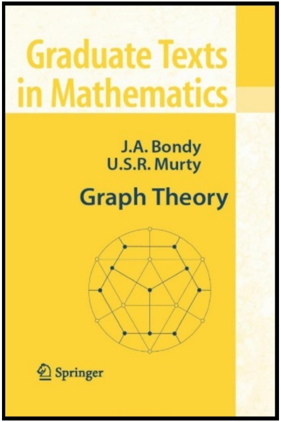
Graph Theory

Chapter 16. Matchings

16.1. Maximum Matchings—Proofs of Theorems



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A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.

Proof. Let M be a matching of G. Suppose that G contains an M-augmenting path P. We claim that $M' = M \triangle E(P)$ is a matching in G. Recall that $A \triangle B = (A \setminus B) \cup (B \setminus A)$, so $M \triangle E(P)$ includes the edges of matching M which are NOT in path P, along with the edges of path P that are NOT in matching M. (We are interchanging the roles of the edges of the M-augmenting path P.)

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Theorem 16.3. Berge's Theorem (continued 1)

Proof (continued). For the converse, suppose that M is not a maximum matching and let M^* be a maximum matching in G so that $|M^*| > |M|$. Let subgraph H of G be the induced subgraph $H = G[M \triangle M^*]$ (see Figure 16.4).

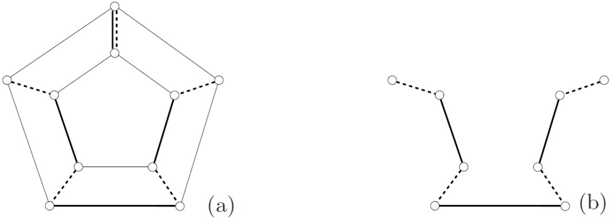

Fig. 16.4. (a) Matchings M (heavy) and M^* (broken), and (b) the subgraph $H := G[M \triangle M^*]$

Each vertex of H (notice that H may not include all vertices of G) has degree one or two in H since it can be incident with at most one edge of M and one edge of M^*.

Theorem 16.3. Berge's Theorem (continued 2)

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Proof (continued). Consequently, each component of H is either an even cycle with edges alternately in M and M^* or a path with edges alternately in M and M^*. Since $|M^*| > |M|$, subgraph H contains more edges of M^* than of M (because $M \triangle M^*$ is a set of edges that results from removing the same number of edges from both M and M^*), and therefore some path-component P of H must start and end with edges of M^*. The origin and terminus of P (here P is a subgraph of H, of course) being covered by M^* are not covered by M. That is, path P is an M-augmenting path in G. Therefore, if is M not a maximum matching of G, then G contains an M-augmenting path, as claimed. □