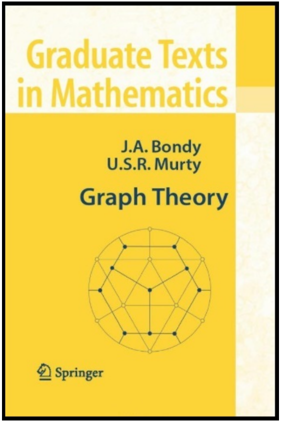
Graph Theory

Chapter 16. Matchings

16.2. Matchings in Bipartite Graphs—Proofs of Theorems



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Proof. Let $G = G[X, Y]$ be a bipartite graph which has a matching M covering every vertex in X. Let $S \subseteq X$. The vertices in S are matched in M with distinct vertices in $N(S)$, defining an injective ("one to one") function from S to $N(S)$. So $|N(S)| \geq |S|$, as claimed.

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Conversely, let $G = G[X, Y]$ be a bipartite graph which has no matching covering every vertex in X. Let M^* be a maximum matching in G and let u be a vertex in X not covered by M^*. Let Z denote the set of all vertices reachable from u by M^*-alternating paths. Because M^* is a maximum matching, by Theorem 16.3 G contains no M^*-augmenting path.

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Theorem 16.4 (continued 1)

Proof (continued). Since $u \in X$ is not covered by M^*, for any $z \in Z$ with $z \neq u$ we have that there is an M^*-alternating path from u to z (which starts with an edge NOT in M^*). If z is not covered by M^*, then the M^*-alternating path from u to z is in fact an M^*-augmented path in G, contradicting the fact that G contains no M^*-augmented paths. Hence, z is covered by M^* and so u is the only element of Z not covered by M^*.

Define $R = X \cap Z$ and $B = Y \cap Z$ (see Figure 16.6).

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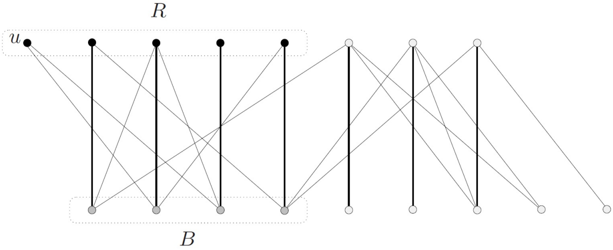

Fig. 16.6. Proof of Hall's Theorem (16.4)

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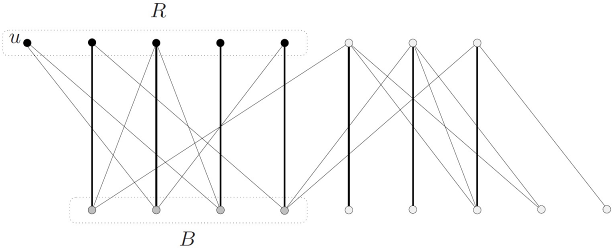

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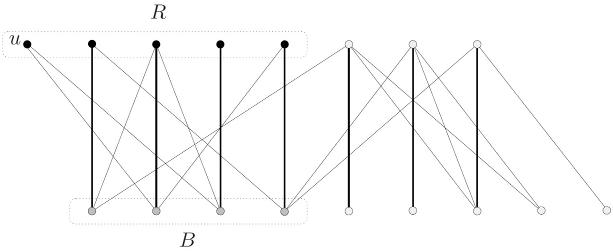

Fig. 16.6. Proof of Hall's Theorem (16.4)

Now the vertices of $R \setminus \{u\}$ are matched under M^* with the vertices of B (because of the M^*-alternating path definition of Z). This implies a bijection between $R \setminus \{u\}$ and B so that $|B| = |R| - 1$. Now the neighbors of vertices in R include all vertices in B; that is $N(R) \supset B$. In fact, every vertex in $N(R) \subset Z$ is connected to u by an M^*-alternating path, and so $N(R) = B$. Hence $|N(R)| = |B| = |R| - 1$.

Theorem 16.4 (continued 3)

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A bipartite graph $G = G[X, Y]$ has a matching which covers every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$ (where $N(S)$ is the set of all vertices which are neighbors of some vertex in S).

Proof (continued). So with set S equal to set R, we have $|N(R)| = |N(S)| < |S| = |R|$. That is, if $G = G[X, Y]$ does not have a matching which covers every vertex in X then $|N(S)| < |S|$ for some $S \subseteq X$, as claimed. □

Corollary 16.6

Corollary 16.6. Every nonempty regular bipartite graph has a perfect matching.

Proof. First, if $G[X, Y]$ is k-regular where $k \geq 1$ then $|X| = |Y|$ by Exercise 1.1.9. Let $S \subseteq X$ and let E_1 and E_2 denote the sets of edges of G incident with S an d$N(S)$, respectively. Notice that $E_1 \subseteq E_2$, since every edge with one end in S must have the other end in $N(S)$ (but not conversely). Since G is k-regular, then $k|N(S)| = |E_2| \geq |E_1| = k|S|$. Therefore $|N(S)| \geq |S|$. Since $S \subseteq X$ is arbitrary, by Corollary 16.5 we have that G has a perfect matching. □

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Theorem 8.32. The König-Egerváry Theorem.
In any bipartite graph G, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

Proof. Let $G = G[X, Y]$ be a bipartite graph with M^* a maximum matching in G, and U the set of vertices in X not covered by M^*. Denote by Z the set of all vertices in G reachable from some vertex in U by M^*-alternating paths.

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Define $R = X \cap Z$ and $B = Y \cap Z$. Let $K^* = (Z \setminus R) \cup B$. Then K^* is a covering with $|K^*| = |M^*|$ by Exercise 16.2.8. By Proposition 16.7, K^* is a minimum cover. That is, $\alpha'(G) = |M^*| = |K^*| = \beta(G)$, as claimed. □

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