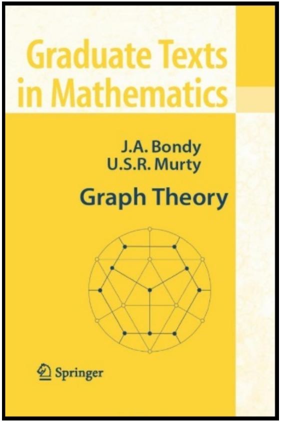
Graph Theory

Chapter 16. Matchings

16.3. Matchings in Arbitrary Graphs—Proofs of Theorems



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$$|U| \geq o(G-S) - |S|. \qquad (16.2)$$

Proof. Let H be an odd connected component of $G-S$ (there are $o(G-S)$ such H). If every vertex of H is covered by M then, since H has an odd number of vertices, at least one vertex of H must be matched with a vertex of S. No more that $|S|$ vertices of $G-S$ can be matched (with respect to M) with vertices of S, so at least $o(G-S)-|S|$ odd components of G must contain vertices not covered by M. Since U is the set of all vertices of G not covered by M, then $|U| \geq o(G-S)-|S|$, as claimed. $\square$

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Lemma 16.10

Lemma 16.10. Let G be a connected graph, no vertex of which is essential. Then G is hypomatchable.

Proof. Since no vertex is essential, then for each vertex v of G then there is a maximum matching of G which does not cover v. So G has no perfect matching. To show that G is hypomatchable, we need to show that every vertex-deleted subgraph of G has a perfect matching. Notice that we can assume that the vertex-deleted graph is on an even number of vertices.

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Lemma 16.10 (continued 1)

Proof (continued). For the base case, suppose the vertices are distance one part; that is, suppose they are adjacent. If neither is covered by the maximum matching, then the edge joining the two vertices can be added to the matching, contradicting maximality. This establishes the base case. For the induction hypothesis, suppose that for any two vertices of G of distance d or less apart, a maximum matching of G contains one or the other vertex.

Consider a maximum matching M and two vertices x and y in G. Let xPy be a shortest xy-path in G and suppose xPy has length $d+1 \geq 2$. ASSUME that neither x nor y is covered by M. Since P has length at least two, there is v an internal vertex of P. Since xPv le length ld or less, then vertex v is covered by M, by the induction hypothesis. By hypothesis v is inessential, so G has a matching M' which does not cover v. Furthermore, because xPv and vPy are both of length d or less, then matching M' covers both x and y, again by the induction hypothesis.

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Lemma 16.10 (continued 2)

Proof (continued). By Exercise 16.1.5, the components of $G[M \triangle M']$ are even length paths whose edges belong alternately to m and M', and even length cycles whose edges belong alternately to M and M'. Each of the vertices x, v, y is covered by exactly one of the two matchings (by construction) and so the edge covering them is in $M \triangle M'$. Now every vertex that is an interior vertex of one of the paths or a vertex of one of the cycles is contained in both an edge of M and an edge of M' (because of the alternating M/M' structure of the paths and cycles). Hence, vertices x, v, y must each be an end vertex of a path. Because the paths are even length, x and y are not ends of the same path (ends of an alternating M/M' even path would have one end in M and the other end in M', but x and y are both covered by M'). Moreover the paths starting at x and y cannot both end at v (since the alternating M/M' property of the even paths would then require both paths to end at v with an edge in M, but two edges in matching M cannot share a vertex).

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Lemma 16.10 (continued 3)

Proof (continued). So we may suppose that the path Q that starts at x ends neither at v nor at y (that is, without loss of generality; if not then the path starting at y has this type of property). Since x is covered by M' (and not covered by M), then even length path Q starts with an edge of M' with s as one end. Denote the other end of Q as w. Since Q is even length and has the M/M' alternating property, then the edge of Q containing w is in M. So no other edge of M can cover w. If an edge of M' contains w then this edge is therefore in $M \triangle M'$. This edge of M' could be used to either increase the length of the M/M' alternating path Q (a contradiction, since Q ends at w) or this edge of M' must be contained in an M/M' alternating cycle (also a contradiction since no other edge of M covers w.

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Lemma 16.10 (continued 4)

Lemma 16.10. Let G be a connected graph, no vertex of which is essential. Then G is hypomatchable.

Proof (continued). Thus $M' \triangle E(Q)$ (the edges of M', but with the edges of M' in Q replaced with the edges of M' in Q) is a matching. Since Q includes $e(Q)/2$ edges of M' and $e(Q)/2$ edges of M, then $|M'| = |M' \triangle E(Q)|$. Since M' is a maximum matching of G, then $M' \triangle E(Q)$ is also a maximum matching of G. However, $M' \triangle E(Q)$ covers neither x nor v. Now we know that the distance between x and v in G is d of less, so the fact that neither x nor v is covered by maximum matching $M' \triangle E(Q)$ CONTRADICTS the induction hypothesis. Hence the assumption that there are two vertices (x and y) of G that are not covered by matching M is false. As described in the initial part of the proof, this establishes the fact that G is hypomatchable, as claimed. □

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