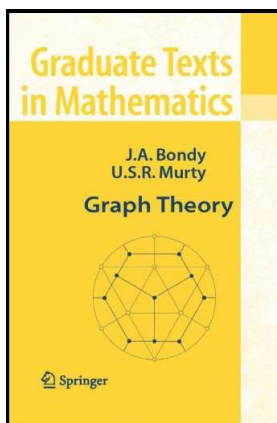


Graph Theory

Chapter 16. Matchings

16.4. Perfect Matchings and Factors—Proofs of Theorems



Theorem 16.13. Tutte's Theorem

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A graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V$.

Proof. By Lemma 16.3.A, we have for any matching M of G that $|U| \geq o(G - S) - |S|$ where U is the set of vertices of G not covered by M and S is any subset of V . For a perfect matching we have $U = \emptyset$, so that $o(G - S) \leq |S|$ for all $S \subseteq V$ so that the equation is a necessary condition for the existence of perfect matching.

Conversely, let G be a graph that has no perfect matching. Consider a maximum matching M^* of G , and denote by U the set of vertices of G not covered by M^* . By the Tutte-Berge Theorem (Theorem 16.11) G has a barrier $B \subseteq V$ where $o(G - B) - |B| = |U|$. Because M^* is not a perfect matching then $|U| \geq 1$. Thus $o(G - B) = |B| + |U| \geq |B| + 1$ so that the equation is violated. We have shown the contrapositive of the converse, so that the claim holds. \square

Theorem 16.14. Petersen's Theorem

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Every 3-regular graph without cut edges has a perfect matching.

Proof. Let G be a 3-regular graph without cut edges. Let $S \subseteq V$. Let the vertex sets of the odd components of $G - S$ be S_1, S_2, \dots, S_k . Recall from Section 2.5. Edge Cuts and Bonds that for $X \subseteq V$ we have $d(X) = |\partial(X)|$ where $\partial(X)$ is the coboundary (or "edge cut") of set X (i.e., the edges of G with exactly one end in X). In G , if $d(S_i) = 1$ then the one edge in $\partial(S_i)$ is a cut edge of G . Since G has no cut edges, then $d(S_i) \geq 2$ for each $1 \leq i \leq k$. Since $|S_i|$ is odd, then by Exercise 2.5.5 $d(S_i)$ is odd for $1 \leq i \leq k$. Therefore, $d(S_i) \geq 3$ for $1 \leq i \leq k$. If $\partial(S_i) \cap \partial(S_j) \neq \emptyset$ for $i \neq j$ then there is an edge e of G with one end in S_i and the other end in S_j . But S_i and S_j are connected components of $G - S$ so e cannot be in $E(G - S)$; also, e cannot be in $E(S)$ or $E[G - S, S]$ (edges with one end in $G - S$ and the other in S) since each end of e is in some S_i (notice that we can include the vertex sets of the even components of $G - S$ in this also, but we are only interested in the odd components).

Theorem 16.14. Petersen's Theorem (continued)

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Every 3-regular graph without cut edges has a perfect matching.

Proof (continued). Therefore the $\partial(S_i)$ are pairwise disjoint and each $\partial(S_i)$ must be contained in $\partial(S)$. We now have

$$3k \leq \sum_{i=1}^k d(S_i) = d\left(\bigcup_{i=1}^k S_i\right) \leq d(S) \leq 3|S|$$

where the last inequality follows from the 3-regular hypothesis. Hence $k \leq |S|$ and so $k = o(G - S) \leq |S|$. Since S is an arbitrary subset of V , by Tutte's Theorem (Theorem 16.13) G has a perfect matching, as claimed. \square