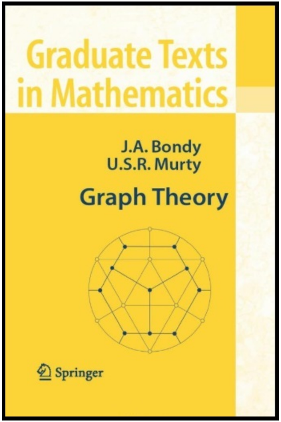
Graph Theory

Chapter 16. Matchings

16.4. Perfect Matchings and Factors—Proofs of Theorems



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Conversely, let G be a graph that has no perfect matching. Consider a maximum matching M^* of G, and denote by U the set of vertices of G not covered by M^*. By the Tutte-Berge Theorem (Theorem 16.11) G has a barrier $B \subseteq V$ where $0(G - B) - |B| = |U|$. Because M^* is not a perfect matching then $|U| \geq 1$. Thus $o(G - B) = |B| + |U| \geq |B| + 1$ so that the equation is violated. We have shown the contrapositive of the converse, so that the claim holds. □

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Theorem 16.14. Petersen's Theorem

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Every 3-regular graph without cut edges has a perfect matching.

Proof. Let G be a 3-regular graph without cut edges. Let $S \subseteq V$. Let the certex sets of the odd components of $G - S$ be $S_1, S_2, \ldots, S_k$. Recall from Section 2.5. Edge Cuts and Bonds that for $X \subseteq V$ we have $d(X) = |\partial(X)|$ where $\partial(X)$ is the coboundary (or "edge cut") of set X (i.e., the edges of G with exactly one end in X). In G, if $d(S_i) = 1$ then the one edge in $\partial(S_i)$ is a cut edge of G. Since G has no cut edges, then $d(S_i) \geq 2$ for each $1 \leq i \leq k$. Since $|S_i|$ is odd, then by Exercise 2.5.5 $d(S_i)$ is odd for $1 \leq i \leq k$.

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Theorem 16.14. Petersen's Theorem (continued)

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Proof (continued). Therefore the $\partial(S_i)$ are pairwise disjoint and each $\partial(S_i)$ must be contained in $\partial(S)$. We now have

$$3k \leq \sum_{i=1}^{k} d(S_i) = d\left(\dot{\bigcup_{i=1}^{k}} S_i\right) \leq d(S) \leq 3|S|$$

where the last inequality follows from the 3-regular hypothesis. Hence $k \leq |S|$ and so $k = o(G - S) \leq |S|$. Since S is an arbitrary subset of V, by Tutte's Theorem (Theorem 16.13) G has a perfect matching, as claimed. □