Graph Theory

Chapter 17. Edge Colourings

17.1. Edge Chromatic Number—Proofs of Theorems

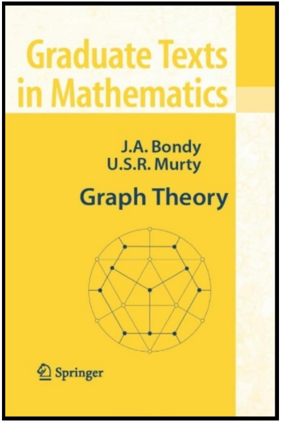

Table of contents

Theorem 17.2

Theorem 17.2. If G is bipartite, then $\chi' = \Delta$.

Proof. We give an inductive proof on the number of edges m. If G has one edge then $\Delta = 1$ and trivially $\chi' = 1$. For the induction hypothesis, suppose that every bipartite graph with $m = k$ edges, where $k \geq 1$, satisfies the theorem.

Theorem 17.2

Theorem 17.2. If G is bipartite, then $\chi' = \Delta$.

Proof. We give an inductive proof on the number of edges m. If G has one edge then $\Delta = 1$ and trivially $\chi' = 1$. For the induction hypothesis, suppose that every bipartite graph with $m = k$ edges, where $k \geq 1$, satisfies the theorem.

Now let G be a graph with $m = k + 1$ edges. Let $e = uv$ be an edge of G. By the induction hypothesis, $H = G \setminus e$ (with $k - 1 = m$ edges) has a Δ-edge-colouring $\{M_1, M_2, \ldots, M_\Delta\}$. If some colour is available for e then that colour can be assigned to e to yield a Δ-edge-colouring of G and the induction step is verified. So we may assume that no colour is available for e, so that each of the colours $1, 2, \ldots, \Delta$ is represented either at u or at v. Since the degree of u in $G \setminus e$ is at most $\Delta - 1$ then at least one colour i is available at u. Since colour i is not available for e then i is represented at v. Similarly, at least one colour $j \neq i$ is available at v and represented at u.

Theorem 17.2

Theorem 17.2. If G is bipartite, then $\chi' = \Delta$.

Proof. We give an inductive proof on the number of edges m. If G has one edge then $\Delta = 1$ and trivially $\chi' = 1$. For the induction hypothesis, suppose that every bipartite graph with $m = k$ edges, where $k \geq 1$, satisfies the theorem.

Now let G be a graph with $m = k + 1$ edges. Let $e = uv$ be an edge of G. By the induction hypothesis, $H = G \setminus e$ (with $k - 1 = m$ edges) has a Δ-edge-colouring $\{M_1, M_2, \ldots, M_\Delta\}$. If some colour is available for e then that colour can be assigned to e to yield a Δ-edge-colouring of G and the induction step is verified. So we may assume that no colour is available for e, so that each of the colours $1, 2, \ldots, \Delta$ is represented either at u or at v. Since the degree of u in $G \setminus e$ is at most $\Delta - 1$ then at least one colour i is available at u. Since colour i is not available for e then i is represented at v. Similarly, at least one colour $j \neq i$ is available at v and represented at u.

Theorem 17.2 (continued)

Proof (continued). Consider the subgraph $H_{ij} = H[M_i \cup M_j]$. Since i is available at u and j is represented at u, then u is of degree one in H_{ij}. By Note/Definition 17.1.B, H_{ij} can be decomposed into even length cycles and paths. So the component of H_{ij} containing u must be an ij-path P. Notice that P cannot terminate at v since this would imply that P is even length since it tarts at u with an edge in E_j and ends at v with an edge in E_i. But $e = uv \in E(G)$ and we could then add edge $e = uv$ to the ij-path creating an odd length cycle in bipartite graph G, an impossibility. So we can interchange the colours i and j on component P of H_{ij} to get an edge colouring of the edges in H_{ij} (though the elements of M_i and M_j are modified) such that colour i is available at both u and v. This gives a Δ-edge-colouring of $G \setminus e$. Since colour i is available at both ends of $e = uv$, then colour i is available for edge e, giving a Δ-edge-colouring of G. Since $\chi' \geq \Delta$, then for bipartite G we have $\chi' = \Delta$. This establishes the induction step, and hence by mathematical induction the result holds for all bipartite graphs. □

Theorem 17.2 (continued)

Proof (continued). Consider the subgraph $H_{ij} = H[M_i \cup M_j]$. Since i is available at u and j is represented at u, then u is of degree one in H_{ij}. By Note/Definition 17.1.B, H_{ij} can be decomposed into even length cycles and paths. So the component of H_{ij} containing u must be an ij-path P. Notice that P cannot terminate at v since this would imply that P is even length since it tarts at u with an edge in E_j and ends at v with an edge in E_i. But $e = uv \in E(G)$ and we could then add edge $e = uv$ to the ij-path creating an odd length cycle in bipartite graph G, an impossibility. So we can interchange the colours i and j on component P of H_{ij} to get an edge colouring of the edges in H_{ij} (though the elements of M_i and M_j are modified) such that colour i is available at both u and v. This gives a Δ-edge-colouring of $G \setminus e$. Since colour i is available at both ends of $e = uv$, then colour i is available for edge e, giving a Δ-edge-colouring of G. Since $\chi' \geq \Delta$, then for bipartite G we have $\chi' = \Delta$. This establishes the induction step, and hence by mathematical induction the result holds for all bipartite graphs. □