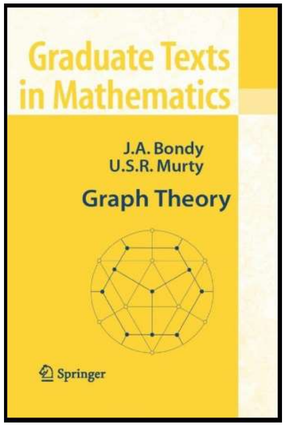
Graph Theory

Chapter 17. Edge Colourings

17.2. Vizing's Theorem—Proofs of Theorems



Lemma 17.3

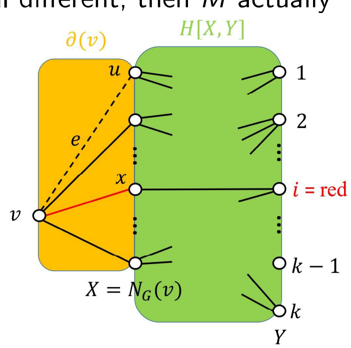
Lemma 17.3. Let G be a simple graph, v a vertex of G, e an edge of G incident to v, and k an integer with $k \geq \Delta$. Suppose that $G \setminus e$ has a k-edge-colouring c with respect to which every neighbor of v has at least one available colour. Then G is k-edge-colourable.

Proof. Consider **FIRST** the k-edge colouring c of $G \setminus e$. With v as the vertex given in the statement of the lemma, let $X = N_G(v)$ and let $Y = \{1, 2, \ldots, k\}$ be the colours in the k-edge-colouring of $G \setminus e$. **SECOND** consider the bipartite graph $H[X, Y]$ where $x \in X$ and $i \in Y$ are adjacent if and only if colour i is available at vertex x in the restriction $\tilde{c}$ of colouring c to $G - v$ (notice that $G - v$ is a subgraph of $G \setminus e$). We now look for a matching of H. For all $x \in X \setminus \{u\}$, where u is the other end of e (i.e., $e = uv$), the colour of the edge xv is available at x in $G - v$ (since edge xv is not in $G - v$). So H includes an edge joining x and the colour of edge xv (denoted $c(xv)$).

Lemma 17.3 (continued 1)

Proof (continued). So H contains the matching $M = \{(x, c(xv)) \mid x \in X \setminus \{u\}\}$; since the original k edge-colouring is proper, the colours $c(xv)$ for $x \in X \setminus \{u\}$ are all different, then M actually is a matching.

Conversely, every matching in H effectively assigns colours to the vertices of X which are covered by the matching. This gives a partial colouring of the coboundary $\partial(v)$ that is compatible with colouring $\tilde{c}$ of $G - v$ (since colours assigned to edges of $\partial(v)$ are based on the available colours for those edges; this is how H is defined).



IF a matching of H saturates X (that is, covers all the vertices of $X = N_G(v)$) then all edges of $\partial(v)$ can be coloured in a way that is compatible with k-edge-colouring $\tilde{c}$ of $G - v$, giving a k-edge-colouring of G, as claimed.

Lemma 17.3 (continued 2)

Proof (continued). We will modify the matching M which covers all vertices of X except for u, to give a new colouring c' which then gives a new bipartite graph H'; we'll show that H' does contain a matching saturating X and the claim then follows.

By hypothesis, there is at least one colour available to each neighbor of v in the k-edge-colouring c of $G \setminus e$, so this gives each vertex $x \in X$ a degree at least one in $H[X, Y]$. *In addition*, the colour $c(xv)$ is available at vertex x in colouring $\tilde{c}$ of $G - v$ (though $c(xv)$ is not available at x in the colouring c of $G \setminus e$). So each vertex of $x \in X \setminus \{u\}$ is of degree at least two in $H[X, Y]$, and so each vertex of $X \setminus \{u\}$ is incident with at least one vertex of $H \setminus M$. Now for vertex $u \in X$, we have

$$\begin{aligned} d_{G \setminus e}(u) &= d_G(u) - 1 \text{ since } e = uv \in E(G) \\ &\leq \Delta(G) - 1 \\ &\leq k - 1 \text{ since } \Delta \leq \chi' \leq k. \end{aligned}$$

Lemma 17.3 (continued 3)

Proof (continued). Hence, u is also incident with at least one vertex of $H \setminus M$ (and by definition of M, u is incident with no edges of M). Therefore, each vertex of X is incident with at least one edge of $H \setminus M$.

We now follow some of the steps of the proof of Hall's Theorem (Theorem 16.4). Denote by Z the set of all vertices of H reachable from u by M-alternating paths (that is, paths that are alternately in M and in $H \setminus M$). Since matching M does not cover u, then each M-alternating path starting at u starts with an edge of $H \setminus M$. Set $R = X \cap Z$ and $B = Y \cap Z$ (so that R is a set of vertices adjacent to v in G and B is a set of colours). The vertices of $R \setminus \{u\}$ are matched under M with the vertices of B (because of the M-alternating path definition of Z). This implies a bijection between $R \setminus \{u\}$ and B so that $|B| = |R| - 1$. Now the neighbors of vertices in R include all vertices in B; that is, $N_H(R) \supseteq B$.

Lemma 17.3 (continued 4)

Proof (continued). In fact, every vertex r in R is connected by an M-alternating path to u (starting at u with an edge in $H \setminus M$ and ending with an edge in M) so if r is joined to a vertex w of H by an edge in $H \setminus M$ then there is an M-alternating path from u to w and so $w \in B$. Notice that this is the only edge of M incident to r (since M is a matching), so the predecessor of r in the M-alternating path from u to r is the only neighbor of r joined to r by an edge of M; of course this neighbor of r is in B. Since r is an arbitrary vertex in R, then $N_H(R) \subseteq B$ and hence $N_H(R) = B$. We now have $|N_H(R)| = |B| = |R| - 1$. Now each vertex of R is incident with at least one edge of $H \setminus M$ (as described above; see slide "Continued 3") so that we have the $|R|$ vertices of R joined to the $|R| - 1$ vertices of $N_H(R)$ by at least $|R|$ edges of $H \setminus M$. Therefore (by the Pigeonhole Principle) some two vertices x and y of R are adjacent in $H \setminus M$ to a common colour $i \in B$. By the definition of bipartite H, this means that colour i is available at both vertices x and y.

Lemma 17.3 (continued 5)

Proof (continued). Notice every colour in B is represented (in G) at vertex v because B is matched (in H) under M with $R \setminus \{u\}$ (see the figure above). In particular, colour i is represented at vertex v. Because the degree of v in $G \setminus e$ is at most $k - 1$ (recall that v is an end of e), some colour $j \neq i$ is available (in $G \setminus e$) at v. Since every colour in $B \subset Y$ is represented at v, then $j \notin B$. If j is available to any vertex $r \in R$ then by the definition of $H = H[X, Y]$, r and j are adjacent in H. But then $j \in N_H(R) = B$ (since $N_H(R) = B$, as shown on slide "Continued 4"), contradicting $j \notin B$. Thus j is not adjacent to (that is, j is represented at) every vertex of R; in particular j is represented at both x and y.

We now exit our discussion of bipartite graph H and return to consideration of graph $G \setminus e$. We are interested in vertices v, x, and y. To summarize, we know that: colour i is represented at v and colour j is available at v, and colour i is available at both x and y and colour j is represented at both x and y.

Lemma 17.3 (continued 6)

Proof (continued). With M_i and M_j as the sets of edges of assigned colours i and j, respectively, define $H_{ij} = H[M_i \cup M_j]$. Notice that $d_{H_{ij}}(v) = d_{H_{ij}}(x) = d_{H_{ij}}(y) = 1$. By Note/Definition 17.1.B, the connected components of H_{ij} are even length cycles and paths. So each of the vertices v, x, and y are ends of path components of H_{ij} (i.e., ends of ij-paths). The ij-path starting at v cannot end at both vertex x and vertex y, so we can assume that it does not end at y (or else, interchange x and y). Let z be the terminal vertex of the ij-path P starting at vertex y (**THIRD**). Next, interchange the colours i and j on P. Since P is a connected component of $H_{ij} = H[M_i \cup M_j]$, then this interchanging of colours causes no conflict of colours (certainly no conflict at interior points of P, but also no conflict at the ends y and z). So this gives **FOURTH** a new colouring c' of $G \setminus e$.

Lemma 17.3 (continued 7)

Proof (continued). **FIFTH**, let $H'[X, Y]$, where $X = N_G(v)$ and $Y = \{1, 2, \ldots, k\}$, be the bipartite graph corresponding to c'; that is, $x \in X$ and $i \in Y$ are adjacent if colour i is available at vertex x in the restriction $\tilde{c}'$ of c' to $G - v$. With respect to colouring $\tilde{c}'$ the same colours are available at each vertex of X except at end vertex y of the ij-path and the end vertex z of the ij-path. However, vertex z may or may not be in $X = N_G(v)$. Therefore the only differences in the edge sets of $H = H[X, Y]$ and $H' = H'[X, Y]$ occur at y and possibly at z (if $z \in X$). Now vertex v does not lie on ij-path P because colour j is not represented at v and v is not an end of the path (the ends are y and z). Thus the colours represented at v in colourings c and c' are the same. Hence the colours available to vertices of $X = N_G(v)$ are the same in colourings $\tilde{c}$ and $\tilde{c}'$ of $G - v$. So matching

$$M = \{(x, c(xv)) \mid x \in X \setminus \{u\}\} = \{(x, c'(vx)) \mid x \in X \setminus \{u\}\}$$

of H is also a matching of H'.

Lemma 17.3 (continued 8)

Proof (continued). Now vertex y is in $R = X \cap Z$ by choice and so (by the definition of Z) y is reachable from u by an M-alternating path Q in H. We use Q to find an M-augmenting path Q' in H' by considering two cases based on the location of vertex z (the terminal vertex of the ij-path P in $G \setminus e$ starting at y).

Case 1. Suppose z lies on Q. Then $z \in X$ (since Q is a path in $H = H[X, Y]$) and $z \in Z$ (since it is then reachable from u by an M-alternating path) and hence $z \in R = X \cap Z$. Since M is a matching in H' also, then uQz is an M-alternating path in H' (since $z \neq y$ then the fact that H' and H differ in that $(y, j) \in E(H)$, $(y, i) \notin E(H)$ but $(y, i) \in E(H')$, $(y, j) \notin E(H')$ is not an issue; at the other end, since $H' = H'[X, Y]$ is bipartite with $u, z \in X$ and uQz starts with an edge not in M, then uQz ends with an edge in M).

Lemma 17.3 (continued 9)

Proof (continued). Since $z \in R$ then, as shown above (see the "Continued 5" slide) j is represented at z in $G \setminus e$. Now i cannot also be represented at z, or else the ij-path P (a connected component of $H_{ij} = H[M_i \cup M_j]$) could be extended beyond z. So the ij-path P from y to z must have originally (that is in colouring c, before the colours i and j were interchanged in P) terminated at z in an edge of colour j. Then, with respect to the colouring c', the colour j is available at z so that (z, j) is an edge of H'. Then $Q' = uQzj$ is an M-augmenting path in H'.

Case 2. Suppose that z does not lie on Q. Since j is represented at y in colouring c (and i is not) then in colour c', i is represented at y (and j is not). That is, colour j is available at vertex y with respect to colouring c' and so (y, j) is an edge of H'. So $Q' = uQyj$ is an M-augmenting path in H'.

In both cases, Q' is an odd length path (it starts at a vertex of X and ends at a vertex of Y).

Lemma 17.3 (continued 10)

Lemma 17.3. Let G be a simple graph, v a vertex of G, e and edge of G incident to v, and k an integer with $k \geq \Delta$. Suppose that $G \setminus e$ has a k-edge-colouring c with respect to which every neighbor of v has at least one available colour. Then G is k-edge-colourable.

Proof (continued). Finally, set $M' = M \triangle E(Q')$ (so that this is the matching M of H', but with the edges of M-augmenting path Q' which are in M replaced with the edges of Q' that are not in M). Since $Q' = uQyj$ is an odd length M-augmenting path starting and ending with edges not in M, this results in a net gain of one edge and includes an edge with u as an end in M' (**SIXTH**). So M' saturates X and, as commented above, this results in a colouring of $\partial(v)$ that is compatible with $\tilde{c}'$, the restriction of c' to $G - v$. This gives the k-edge-colouring of G, as desired. □

Theorem 17.4. Vizing's Theorem

Theorem 17.4. Vizing's Theorem.
For any simple graph G, $\chi' \leq \Delta + 1$.

Proof. We give an inductive proof on the number of edges m of G. For the base case $m = 1$, G has one edge, and $\Delta = 1$. So G has a 1-edge colouring and hence $1 = \chi' \leq \Delta + 1 = 2$. For the induction hypothesis, suppose that all simple graphs on $m = \ell$ edges have chromatic number $\chi' \leq \Delta + 1$.

Now consider G a simple graph on $m = \ell + 1$ edges. For e any edge of G, we have that $G \setminus e$ is a graph on $m = (\ell + 1) - 1 = \ell$ edges. Let v be one end of e. By the induction hypothesis, $\chi'(G \setminus e) \leq \Delta(G \setminus e) + 1$. Since $\Delta(G \setminus e) \leq \Delta(G)$, we have $\chi'(G \setminus e) \leq \Delta(G) + 1$. With $k = \Delta(G) + 1$, we then have (by the definition of $\chi'(G \setminus e)$) that there is a k-edge-colouring of $G \setminus e$. Since $k = \Delta(G) + 1 \geq \Delta(G \setminus e) + 1 > d_{G\setminus}(u)$ for all $u \in V(G \setminus e)$, then each vertex of $G \setminus e$ has at least one available colour.

Theorem 17.4. Vizing's Theorem (continued)

Theorem 17.4. Vizing's Theorem.
For any simple graph G, $\chi' \leq \Delta + 1$.

Proof. In particular, each neighbor of v in G has at least one available colour. By Lemma 17.3, G has a k-edge-colouring. Therefore $\chi'(G) \leq k = \Delta(G) + 1$. This establishes the induction step. Therefore, by mathematical induction, the claim holds for all simple graphs. □