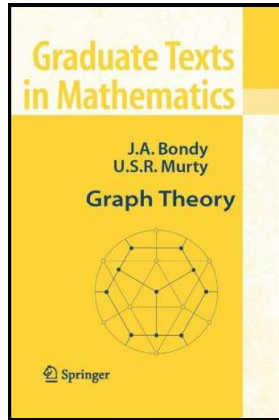


# Graph Theory

## Chapter 17. Edge Colourings

### 17.5. List Edge Colourings—Proofs of Theorems



## Theorem 17.9

**Theorem 17.9.** Let  $G[X, Y]$  be a simple bipartite graph, and let  $D$  be an orientation of its line graph  $L(G)$  in which each  $X$ -clique and each  $Y$ -clique induces a transitive tournament. Then  $D$  has a kernel.

**Proof.** We give an inductive proof on the number of edges of  $G$ ,  $e(G)$ . For  $e(G) = 1$ , we have  $L(G) = K_1$  and each  $X$ -clique and  $Y$ -clique consists of a single vertex and the result holds trivially.

Before we state the induction hypothesis, we introduce some notation. For  $v \in V(G)$ , denote by  $T_v$  the transitive tournament in  $D$  corresponding to  $v$  (see Figure 17.7 in the notes). Every transitive tournament has a sink and a source (by Exercise 2.2.A; this is where we need transitivity). For  $x \in X$  let  $t_x$  be the sink in transitive tournament  $T_x$ . Define set  $K = \{t_x \mid x \in X\}$  (in Figure 17.8 below this would be vertices  $x_1y_3, x_2y_4, x_3y_2$ ).

## Theorem 17.9 (continued 1)

**Proof (continued).**

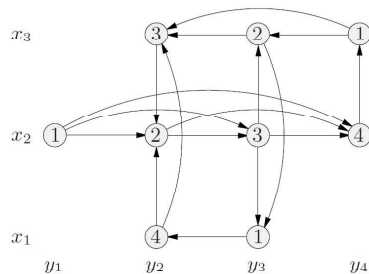


Fig. 17.8. Orienting the line graph of a bipartite graph

Every vertex of  $D - K$  lies in some  $T_x$  (since the vertex sets of  $X$ -cliques partition the vertex set of  $L(G)$ ) and so every vertex of  $D - K$  dominates some vertex of  $K$ . If set  $K$  of vertices of  $D$  is a stable set, then  $K$  is a kernel of  $D$ . If the vertices of  $K$  lie in distinct  $Y$ -cliques then they form a stable set and hence a kernel of  $D$ .

## Theorem 17.9 (continued 2)

**Proof (continued).** Notice that, by choice, the elements of  $K$  are from different  $X$ -cliques; so no two vertices of  $K$  lie in either the same row or the same column of the grid representing  $L(G)$ , see Figure 17.7, and so are not adjacent.

Next, suppose the result holds for all simple bipartite graphs on  $m$  edges (where  $m \geq 1$ ). This is the induction hypothesis. Let  $G[X, Y]$  be a simple bipartite graph on  $e(G) = m + 1$  edges. Since the line graph  $L(G)$  is a union of  $X$ -cliques and  $Y$ -cliques and the orientation  $D$  of  $L(G)$  is a union of transitive tournaments, then the result will follow if all the elements of  $K = \{t_x \mid x \in X\}$  lie in different  $Y$ -cliques. Suppose, then, that the  $Y$ -clique  $T_y$  contains two vertices of  $K$  (we will create a kernel of  $D$ ). One of these vertices, say it is vertex  $t_x$ , is not the source  $s_y$  of  $T_y$  (since we've assumed there are two elements of  $K$  in  $T_y$ ). So  $s_y$  dominates  $t_x$  (that is,  $(s_y, t_x)$  is an arc of  $D$ ). Set  $D' = D - s_y$ .

## Theorem 17.9 (continued 3)

**Proof (continued).** Then  $D'$  is an orientation of the line graph  $L(G \setminus e)$ , where  $e$  is the edge of  $G$  corresponding to the vertex  $s_y$  of  $L(G)$ . Now  $G \setminus e$  is a simple bipartite graph on  $m$  edges and each clique of  $D'$  induces a transitive tournament (in fact, all cliques of  $D$  and  $D'$  are the same, except for the  $Y$ -clique  $T_y$  of  $D$  which corresponds to the  $Y$ -clique  $T_y \setminus \{s_y\}$  of  $D'$  with one less vertex). By the induction hypothesis,  $D'$  has a kernel  $K'$ . We next show that  $K'$  is also a kernel of  $D$ . For this, it suffices (by the definition of kernel) to show that  $s_y$  dominates some vertex of  $K'$ , which we now do.

If  $t_x \in K'$  then  $s_y$  dominates  $t_x$  in  $D$ , since  $(s_y, t_x)$  is an arc of  $D$ , and we are done. If  $t_x \notin K'$ , then  $t_x$  dominates  $v$  for some  $v \in K'$  because then  $t_x \in D' - K'$  and  $K'$  is a kernel of  $D'$ , so that  $(t_x, v)$  is an arc of  $D'$ . Since  $t_x$  is the sink of its  $X$ -clique (by definition of  $t_x$ ) then  $v$  cannot lie in the  $X$ -clique containing  $t_x$  and so  $v$  must lie in the  $Y$ -clique of  $D'$  containing  $t_x$ .

## Theorem 17.9 (continued 4)

**Theorem 17.9.** Let  $G[X, Y]$  be a simple bipartite graph, and let  $D$  be an orientation of its line graph  $L(G)$  in which each  $X$ -clique and each  $Y$ -clique induces a transitive tournament. Then  $D$  has a kernel.

**Proof (continued).** (Since  $E'$  contains only arcs within transitive tournaments  $T_u$  where  $u$  is an element of  $X$  or an element of  $Y$ ; that is, in the grid representation of the line graph, there are only edges between vertices in the same row or the same column, as discussed in Note 17.5.C). Since  $t_x$  was chosen from  $Y$ -clique  $T_y$  of  $D$ , then we must have  $v$  in  $Y$ -clique  $T_y - \{s_y\}$  of  $D'$ . Now  $s_y$  is the source of  $T_y$  in  $D$  and so  $s_y$  dominates  $v$ . In both cases (namely,  $t_x \in K'$  and  $t_x \notin K'$ ), we have that  $s_y$  dominates some element of  $K'$ . As described above, this shows that  $K'$  is a kernel of  $D$ . This established the induction step. Therefore, by mathematical induction, the result holds for all simple bipartite graphs.  $\square$

## Theorem 17.10

**Theorem 17.10.** Every simple bipartite graph  $G$  is  $\Delta$ -list-edge-colourable.

**Proof.** Let  $G = G[X, Y]$  be a simple bipartite graph with maximum degree  $\Delta = k$ . Let  $c : E(G) \rightarrow \{a, 2, \dots, k\}$  be a  $k$ -edge-colouring of  $G$  which exists by Theorem 17.2. The colouring  $c$  induces a  $k$ -colouring of  $L(G)$  by Note 17.5.B. Orient each edge of  $L(G)$  joining two vertices of an  $X$ -clique from lower to higher colour, and orient each edge of  $L(G)$  joining two vertices of a  $Y$ -clique from higher to lower colour, as in Figure 17.8 (above). Call this orientation of  $L(G)$  digraph  $D$ . The 4-edge-colouring of  $G$  is indicated in the line graph of Figure 17.8 with the labels on the vertices; the  $X$ -cliques are “horizontal” and the  $Y$ -cliques are “vertical” here (see Note 17.5.C). The orientations on the  $X$ -cliques and  $Y$ -cliques are based on the vertex colouring of  $L(G)$  and so each is a transitive subgraph of  $L(G)$ ; that is, each is a transitive tournament. So the hypotheses of Theorem 17.9 are satisfied.

## Theorem 17.10 (continued)

**Theorem 17.10.** Every simple bipartite graph  $G$  is  $\Delta$ -list-edge-colourable.

**Proof (continued).** In fact, any induced subgraph of  $D$  is either an  $X$ -clique, a  $Y$ -clique, or has connected components that are intersections of  $X$ -cliques and  $Y$ -cliques. Such subgraphs also satisfy the hypotheses of Theorem 17.9. Hence by Theorem 17.9, every subgraph of  $D$  has a kernel. Moreover the maximum outdegree in  $D$  is  $\Delta^+(D) = k - 1$  (a vertex of  $G$  of degree  $\Delta$  is in an  $X$ -clique or a  $Y$ -clique of  $k$  vertices and so this clique contains a vertex of outdegree  $k - 1$  (the vertex of lowest colour in the clique if it is an  $X$ -clique and the vertex of highest colour in the clique if it is a  $Y$ -clique). So by Theorem 14.20,  $L(G)$  is  $((k - 1) + 1)$ -list colourable (i.e.,  $k$ -list colourable). Therefore, by Note 17.5.B,  $G$  is list-edge-colourable, as claimed.  $\square$