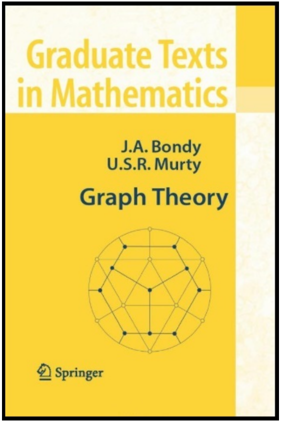
Graph Theory

Chapter 17. Edge Colourings

17.5. List Edge Colourings—Proofs of Theorems



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Proof. We give an inductive proof on the number of edges of G, $e(G)$. For $e(G) = 1$, we have $L(G) = K_1$ and each X-clique and Y-clique consists of a single vertex and the result holds trivially.

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Before we state the induction hypothesis, we introduce some notation. For $v \in V(G)$, denote by T_v the transitive tournament in D corresponding to v (see Figure 17.7 in the notes). Every transitive tournament has a sink and a source (by Exercise 2.2.A; this is where we need transitivity). For $x \in X$ let t_x be the sink in transitive tournament T_v. Define set $K = \{t_x \mid x \in X\}$ (in Figure 17.8 below this would be vertices x_1y_3, x_2y_4, x_3y_2).

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Theorem 17.9 (continued 1)

Proof (continued).

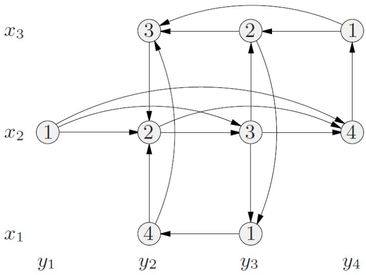

Fig. 17.8. Orienting the line graph of a bipartite graph

Every vertex of $D - K$ lies in some T_x (since the vertex sets of X-cliques partition the vertex set of $L(G)$) and so every vertex of $D - K$ dominates some vertex of K. If set K of vertices of D is a stable set, then K is a kernel of D. If the vertices of K lie in distinct Y-cliques then they form a stable set and hence a kernel of D.

Theorem 17.9 (continued 2)

Proof (continued). Notice that, by choice, the elements of K are from different X-cliques; so no two vertices of K lie in either the same row or the same column of the grid representing $L(G)$, see Figure 17.7, and so are not adjacent.

Next, suppose the result holds for all simple bipartite graphs on m edges (whee $m \geq 1$). This is the induction hypothesis. Let $G[X, Y]$ be a simple bipartite graph on $e(G) = m + 1$ edges. Since the line graph $L(G)$ is a union of X-cliques and Y-cliques and the orientation D of $L(G)$ is a union of transitive tournaments, then the result will follow if all the elements of $K = \{t_x \mid x \in X\}$ lie in different Y-cliques. Suppose, then, that the Y-clique T_y contains two vertices of K (we will create a kernel of D). One of these vertices, say it is vertex t_x, is nto the source s_y of T_y (since we've assumed there are *two* elements of K in T_y). So s_y dominates t_x (that is, (s_y, t_x) is an arc of D). Set $D' = D - s_y$.

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Theorem 17.9 (continued 3)

Proof (continued). Then D' is an orientation of the line graph $L(G \setminus e)$, where e is the edge of G corresponding to the vertex s_y of $L(G)$. Now $G \setminus e$ is a simple bipartite graph on m edges and each clique of D' induces a transitive tournament (in fact, all cliques of D and D' are the same, except for the Y-clique T_y of D which corresponds to the Y-clique $T_y \setminus \{s_y\}$ of D' with one less vertex). By the induction hypothesis, D' has a kernel K'. We next show that K' is also a kernel of D. For this, it suffices (by the definition of kernel) to show that s_y dominates some vertex of K', which we now do.

If $t_x \in K'$ then x_y dominates t_x in D, since $s_y, t_x)$ is an arc of D, and we are done. If $t_x \notin K'$, then t_x dominates v for some $v \in K'$ because then $t_x \in D' - K'$ and K' is a kernel of D', so that (t_x, v) is an arc of D'. Since t_x is the sink of its X-clique (by definition of t_x) then v cannot lie in the X-clique containing t_x and so v must lie in the Y-clique of D' containing t_x.

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If $t_x \in K'$ then x_y dominates t_x in D, since $s_y, t_x)$ is an arc of D, and we are done. If $t_x \notin K'$, then t_x dominates v for some $v \in K'$ because then $t_x \in D' - K'$ and K' is a kernel of D', so that (t_x, v) is an arc of D'. Since t_x is the sink of its X-clique (by definition of t_x) then v cannot lie in the X-clique containing t_x and so v must lie in the Y-clique of D' containing t_x.

Theorem 17.9 (continued 4)

Theorem 17.9. Let $G[X, Y]$ be a simple bipartite graph, and let D be an orientation of its line graph $L(G)$ in which each X-clique and each Y-clique induces a transitive tournament. Then D has a kernel.

Proof (continued). (Since E' contains only arcs within transitive tournaments T_u where u is an element of X or an element of Y; that is, in the grid representation of the line graph, there are only edges between vertices in the same row or the same column, as discussed in Note 17.5.C). Since t_x was chosen from Y-clique T_y of D, then we must have v in Y-clique $T_y - \{s_y\}$ of D'. Now s_y is the source of T_y in D and so s_y dominates v. In both cases (namely, $t_x \in K'$ and $t_x \notin K'$), we have that S_y dominates some element of K'. As described above, this shows that K' is a kernel of D. This established the induction step. Therefore, by mathematical induction, the result holds for all simple bipartite graphs. □

Theorem 17.10

Theorem 17.10. Every simple bipartite graph G is Δ-list-edge-colourable.

Proof. Let $G = G[X, Y]$ be a simple bipartite graph with maximum degree $\Delta = k$. Let $c : E(G) \to \{a, 2, \ldots, k\}$ be a k-edge-colouring of G which exists by Theorem 17.2. The colouring c induces a k-colouring of $L(G)$ by Note 17.5.B. Orient each edge of $L(G)$ joining two vertices of an X-clique from lower to higher colour, and orient each edge of $L(G)$ joining two vertices of a Y-clique from higher to lower colour, as in Figure 17.8 (above). Call this orientation of $L(G)$ digraph D.

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Theorem 17.10 (continued)

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Proof (continued). In fact, any induced subgraph of D is either an X-clique, a Y-clique, or has connected components that are intersections of X-cliques and Y-cliques. Such subgraphs also satisfy the hypotheses of Theorem 17.9. Hence by Theorem 17.9, every subgraph of D has a kernel. Moreover the maximum outdegree in D is $\Delta^+(D) = k - 1$ (a vertex of G of degree Δ is in an X-clique or a Y-clique of k vertices and so this clique contains a vertex of outdegree $k - 1$ (the vertex of lowest colour in the clique if it is an X-clique and the vertex of highest colour in the clique if it is a Y-clique). So by Theorem 14.20, $L(G)$ is $((k - 1) + 1)$-list colourable (i.e., k-list colourable). Therefore, by Note 17.5.B, G is lest-edge-colourable, as claimed. □

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