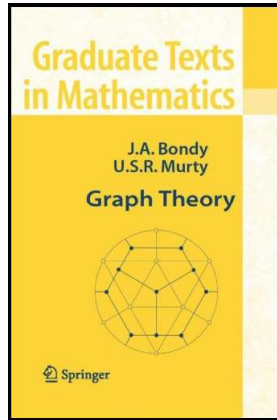


Graph Theory

Chapter 17. Edge Colourings

17.6. Further Reading—Proofs of Theorems



Theorem 17.6.A

Theorem 17.6.A. For $m \neq n$, $\chi''(K_{m,n}) = \Delta + 1$. In addition, $\chi''(K_{n,n}) = \Delta + 2$.

Proof. Suppose $m \neq n$. Say, without loss of generality, $m < n$. Let $X = \{x_0, x_1, \dots, x_{m-1}\}$, $Y = \{y_0, y_1, \dots, y_{n-1}\}$, and $K_{m,n} = G[X, Y]$. Define $A_p = \{x_i y_{i+p} \mid i = 0, 1, \dots, m-1\}$ for $p = 0, 1, \dots, n-1$ where the indices $i+p$ are reduced modulo n . Notice that each A_p for $p = 0, 1, \dots, n-1$ contains no two edges incident to the same vertex. So for a given p the same colour can be assigned to all edges in A_p . (By the way, the sets of edges A_p for $p = 0, 1, \dots, n-1$ partition the edge set of $K_{m,n}$ and this show that the edge chromatic number of $K_{m,n}$ is $\chi'(K_{m,n}) = n = \Delta(K_{m,n})$, so that $K_{m,n}$ is Class 1.) Notice for $i \in \{0, 1, \dots, m-1\}$ we have $(i+p) \pmod n \neq p-1$ (in particular, $((m-1)+p) \pmod n = ((p-1) + m \pmod n) < p-1$ because $m < n$).

Theorem 17.6.A (continued 1)

Proof (continued). So no edge of A_p is incident with vertex y_{p-1} , and the colour assigned to the edges of A_p can also be assigned to vertex y_{p-1} for $p = 0, 1, \dots, n-1$. This gives a proper total colouring (with n colours) of all edges of $K_{n,n}$ and all vertices of set $Y = \{y_0, y_1, \dots, y_n\}$. Finally, we use one additional colour on the vertices of set $X = \{x_0, x_1, \dots, x_{m-1}\}$. This gives a total colouring of $K_{m,n}$ with $n+1 = \Delta+1$ colours. Therefore $\chi''(K_{m,n}) \leq \Delta+1$. By Note 17.6.A we have $\chi''(G) \geq \Delta+1$ and hence $\chi''(K_{m,n}) = \Delta+1$, as claimed.

Now consider $K_{n,n}$. Let S be a set of vertices and edges of $K_{n,n}$ that are all assigned the same colour in a proper total colouring of $K_{n,n}$. ASSUME $|S| > n$. Notice that S cannot contain all edges (since this would imply that S contains two edges incident to the same vertex) nor can S contain all vertices (since this would imply that S contains two adjacent vertices). Let ν be the number of vertices of S and ε be the number of edges of S , so that $|S| = \nu + \varepsilon$.

Theorem 17.6.A (continued 2)

Proof (continued). Each vertex of S is adjacent to n other vertices and incident to n edges; each edge of S is incident to 2 vertices and shares a vertex with $2(n-1)$ other edges. Now the number of elements of S plus the number of edges and arcs incident/adjacent to an element of S cannot exceed the total number of edges and vertices. That is, $(\nu + \varepsilon) + \nu(n + n) + \varepsilon(2 + 2(n-1)) \leq n^2 + 2n$ or $(\nu + \varepsilon) + 2n\nu + 2n\varepsilon \leq n^2 + 2n$ or $(\nu + \varepsilon)(2n+1) \leq n^2 + 2n$. However, we have assumed that $|S| = \nu + \varepsilon > n$ and we must have $(\nu + \varepsilon)(2n+1) > n(2n+1) = 2n^2 + n = n^2 + (n^2 + n) \geq n^2 + (n+n) = n^2 + 2n$, a CONTRADICTION. So the assumption that $|S| > n$ is false, and hence $|S| \leq n$. So the largest that a "colour class" of a total colouring of $K_{n,n}$ can be is n . Since $|E(K_{n,n})| + |V(K_{n,n})| = n^2 + 2n$, then a total colouring of all edges and vertices of $K_{n,n}$ requires at least $(n^2 + 2n)/n = n + 2$ colours. Therefore $\chi''(K_{n,n}) \geq n + 2 = \Delta + 2$.

Theorem 17.6.A (continued 3)

Theorem 17.6.A. For $m \neq n$, $\chi''(M_{m,n}) = \Delta + 1$. In addition, $\chi''(K_{n,n}) = \Delta + 2$.

Proof (continued). With A_p defined as in the case $m \neq n$ (above) for $p = 0, 1, \dots, n - 1$, we can assign a different colour to each of these sets of edges (using n colours). We can colour set X of vertices with another colour, and set Y of vertices with a final colour. This gives a total colouring of $K_{n,n}$ with $n + 2 = \Delta + 2$ colours so that $\chi''(K_{n,n}) \leq \Delta + 2$. Since we have $\chi''(K_{n,n}) \geq \Delta + 2$ from above, we can conclude that $\chi''(K_{n,n}) = \Delta + 2$, as claimed. \square