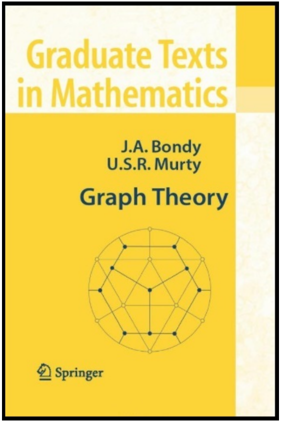
Graph Theory

Chapter 17. Edge Colourings

17.6. Further Reading—Proofs of Theorems



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Theorem 17.6.A

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Proof. Suppose $m \neq n$. Say, without loss of generality, $m < n$. Let $X = \{x_0, x_1, \ldots, x_{m-1}\}$ $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, and $K_{m,n} = G[X, Y]$. Define $A_p = \{x_i y_{i+p} \mid i = 0, 1, \ldots, m-1\}$ for $p = 0, 1, \ldots, n-1$ where the indices $i + p$ are reduced modulo n. Notice that each A_p for $p = 0, 1, \ldots, n-1$ contains no two edges incident to the same vertex. So

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Theorem 17.6.A (continued 1)

Proof (continued). So no edge of A_p is incident with vertex y_{p-1}, and the colour assigned to the edges of A_p can also be assigned to vertex y_{p-1} for $p = 0, 1, \ldots, n-1$. This gives a proper total colouring (with n colours) of all edges of $K_{n,n}$ and all vertices of set $Y = \{t_0, y_1, \ldots, y_n\}$. Finally, we use one additional colour on the vertices of set $X = \{x_0, x_1, \ldots, x_{m-1}\}$. This gives a total colouring of $K_{m,n}$ with $n+1 = \Delta + 1$ colours. Therefore $\chi''(K_{m,n}) \leq \Delta + 1$. By Note 17.6.A we have $\chi''(G) \geq \Delta + 1$ and hence $\chi''(K_{m,n}) = \Delta + 1$, as claimed.

Now consider $K_{n,n}$. Let S be a set of vertices *and* edges of $K_{n,n}$ that are all assigned the same colour in a proper total colouring of $K_{n,n}$. ASSUME $|S| > n$. Notice that S cannot contain all edges (since this would imply that S contains two edges incident to the same vertex) nor can S contain all vertices (since this would imply that S contains two adjacent vertices). Let ν be the number of vertices of S and ε be the number of edges of S, so that $|S| = \nu + \varepsilon$.

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Theorem 17.6.A (continued 2)

Proof (continued). Each vertex of S is adjacent to n other vertices and incident to n edges; each edge of S is incident to 2 vertices and shares a vertex with $2(n-1)$ other edges. Now the number of elements of S plus the number of edges and arcs incident/adjacent to an element of S cannot exceed the total number of edges and vertices. That is,
$(\nu+\varepsilon)+\nu(n+n)+\varepsilon(2+2(n-1)) \le n^2+2n$ or
$(\nu+\varepsilon)+2n\nu+2n\varepsilon \le n^2+2n$ or $(\nu+\varepsilon)(2n+1) \le n^2+2n$. However, we have assumed that $|S| = \nu+\varepsilon > n$ and we must have
$(\nu+\varepsilon)(2n+1) > n(2n+1) = 2n^2+n = n^2+(n^2+n) \ge n^2+(n+n) = n^2+2n$,
a CONTRADICTION. So the assumption that $|S| > n$ is false, and hence $|S| \le n$. So the largest that a "colour class" of a total colouring of $K_{n,n}$ can be is n. Since $|E(K_{n,n})| + |V(K_{n,n})| = n^2+2n$, then a total coluring of all edges and vertices of $K_{n,n}$ requires at least $(n^2+2n)/n = n+2$ colours. Therefore $\chi''(K_{n,n}) \ge n+2 = \Delta+2$.

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Theorem 17.6.A (continued 3)

Theorem 17.6.A. For $m \neq n$, $\chi''(M_{m,n}) = \Delta + 1$. In addition, $\chi''(K_{n,n}) = \Delta + 2$.

Proof (continued). With A_p defined as in the case $m \neq n$ (above) for $p = 0, 1, \ldots, n-1$, we can assign a different colour to each of these sets of edges (using n colours). We can colour set X of vertices with another colour, and set Y of vertices with a final colour. This gives a total colouring of $K_{n,n}$ with $n + 2 = \Delta + 2$ colours so that $\chi''(K_{n,n}) \leq \Delta + 2$. Since we have $\chi''(K_{n,n}) \geq \Delta + 2$ from above, we can conclude that $\chi''(K_{n,n}) = \Delta + 2$, as claimed. □