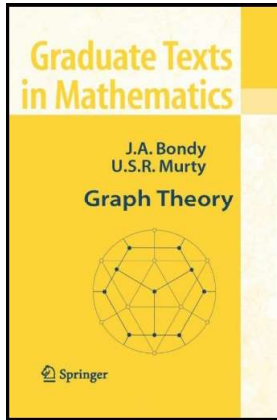


Graph Theory

Chapter 2. Subgraphs

2.1. Subgraphs and Supergraphs—Proofs of Theorems



Theorem 2.1

Theorem 2.1. Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.

Proof. If G is not simple then it either contains a loop (i.e., a cycle of length one) or parallel edges (two of which with the same ends form a cycle of length two). So we can assume without loss of generality that G is simple.

Let $P = v_0 v_1 \cdots v_{k-1} v_k$ be a path in G of longest length (which exists since G is finite). Since vertex v_k is of degree at least two by hypothesis then it has a neighbor v different from v_{k-1} . If v is not on P , then the path $v_0 v_1 \cdots v_k v$ is larger than path P , contradicting the choice of P . So it must be that $v = v_i$ for some $0 \leq i \leq k - 2$. Then G contains the cycle $v_i v_{i+1} \cdots v_k v_i$. \square

Theorem 2.2

Theorem 2.2. Any simple graph G with $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$ contains a quadrilateral.

Proof. Denote by p_2 the number of distinct paths of length 2 in G . Denote by $p_2(v)$ the number of such paths whose “central” vertex is v . Now for a given vertex v where $d(v) \geq 2$, we can choose 2-paths with v as the central vertex in $\binom{d(v)}{2}$ different ways.

So for each vertex v we have $p_2(v) = \binom{d(v)}{2}$ (where we interpret $\binom{0}{2} = \binom{1}{2} = 0$). Since each 2-path has a unique central vertex, then

$$p_2 = \sum_{v \in V} p_2(v) = \sum_{v \in V} \binom{d(v)}{2}. \quad (*)$$

Next, each 2-path has a unique pair of end vertices. So we can create sets of 2-paths where two 2-paths are in the same set if they have the same end vertices.

Theorem 2.2 (continued)

Theorem 2.2. Any simple graph G with $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$ contains a quadrilateral.

Proof (continued). There are then $\binom{n}{2}$ such sets (though some could be empty). We have hypothesized $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$. So by $(*)$, $p_2 > \binom{n}{2}$ and by the Pigeonhole Principle one of the sets of 2-paths with common end vertices must contain at least two 2-paths. Since these two 2-paths have different central vertices (because they are elements of a set) then the union of these two 2-paths forms a quadrilateral that is contained in G , as claimed. \square