Graph Theory

Chapter 2. Subgraphs

2.1. Subgraphs and Supergraphs—Proofs of Theorems

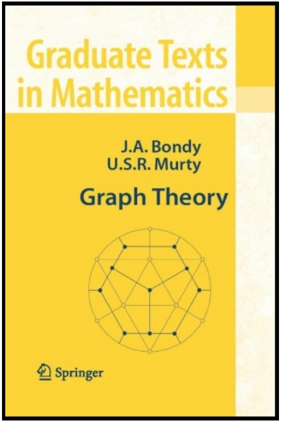

Table of contents

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Let $P = v_0 v_1 \cdots v_{k-1} v_k$ be a path in G of longest length (which exists since G is finite). Since vertex v_k is of degree at least two by hypothesis then it has a neighbor v different from v_{k-1}. If v is not on P, then the path $v_0 v_1 \cdots v_k v$ is larger than path P, contradicting the choice of P. So it must be that $v = v_i$ for some $0 \leq i \leq k-2$. Then G contains the cycle $v_i v_{i+1} \cdots v_k v_i$. □

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Theorem 2.2

Theorem 2.2. Any simple graph G with $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$ contains a quadrilateral.

Proof. Denote by p_2 the number of distinct paths of length 2 in G. Denote by $p_2(v)$ the number of such paths whose "central" vertex is v. Now for a given vertex v where $d(v) \geq 2$, we can choose 2-paths with v as the central vertex in $\binom{d(v)}{2}$ different ways.

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So for each vertex v we have $p_2(v) = \binom{d(v)}{2}$ (where we interpret $\binom{0}{2} = \binom{1}{2} = 0$). Since each 2-path has a unique central vertex, then

$$p_2 = \sum_{v \in V} p_2(v) = \sum_{v \in V} \binom{d(v)}{2}. \qquad (*)$$

Next, each 2-path has a unique pair of end vertices. So we can create sets of 2-paths where two 2-paths are in the same set if they have the same end vertices.

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Theorem 2.2 (continued)

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Proof (continued). There are then $\binom{n}{2}$ such sets (though some could be empty). We have hypothesized $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$. So by $(*)$, $p_2 > \binom{n}{2}$ and by the Pigeonhole Principle one of the sets of 2-paths with common end vertices must contain at least two 2-paths. Since these two 2-paths have different central vertices (because they are elements of a *set*) then the union of these two 2-paths forms a quadrilateral that is contained in G, as claimed. □