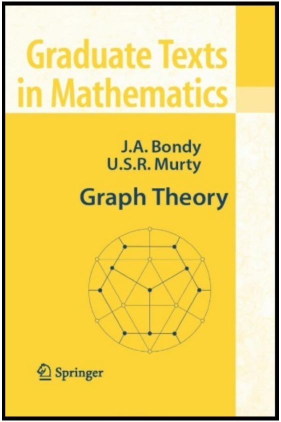
Graph Theory

Chapter 2. Subgraphs

2.2. Spanning and Induced Subgraphs—Proofs of Theorems



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Proof. We give an inductive proof. The trivial tournament (on one vertex) has a directed Hamilton path (of length 0), so the result holds for a tournament of order 1. Hypothesize that for some integer $n \geq 2$, every tournament on $n-1$ vertices has a directed Hamilton path (this is the induction hypothesis). Let T be a tournament on n vertices and let $v \in V(T)$. The digraph $T' = T - v$ is a tournament on $n-1$ vertices. By the induction hypothesis, T' has a directed Hamilton path, say $P' = (v_1, v_2, \ldots, v_{n-1})$. We now go through three cases.

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(1) If (v, v_1) is an arc of T, the path $(v, v_1, v_2, \ldots, v_{n-1})$ is a directed Hamilton path of T.

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(1) If (v, v_1) is an arc of T, the path $(v, v_1, v_2, \ldots, v_{n-1})$ is a directed Hamilton path of T.

(2) If (v_{n-1}, v) is an arc of T, the path $(v_1, v_2, \ldots, v_{n-1}, v)$ is a directed Hamilton path of T.

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(2) If (v_{n-1}, v) is an arc of T, the path $(v_1, v_2, \ldots, v_{n-1}, v)$ is a directed Hamilton path of T.

Theorem 2.3 (continued 1)

Proof (continued).

(3) If neither (v, v_1) nor (v_{n-1}, v) is an arc of T then (since T is a tournament; i.e., an orientation of K_n) then both (v_1, v) and (v, v_{n-1}) must be arcs of T. That is, there is an arc from path P' to vertex v with tail v_1, and there is an arc from vertex v to path P' with head v_{n-1}. For each of $v_2, v_3, \ldots, v_{n-2}$, there is either an arc from v to v_i or from v_i to v (but not both) for $i = 2, 3, \ldots, n-2$. Since arc (v_1, v) goes from P' to v and arc (v, v_{n-1}) goes from v to P', then there must be some $i \in \{1, 2, \ldots, n-2\}$ such that (v_i, v) and (v, v_{i+1}) are arcs of T (where the arcs "change" from going from P' to going to P'):

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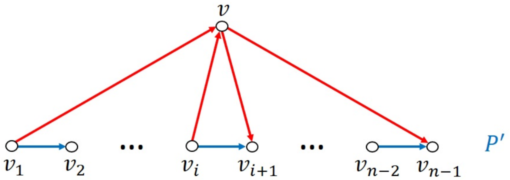

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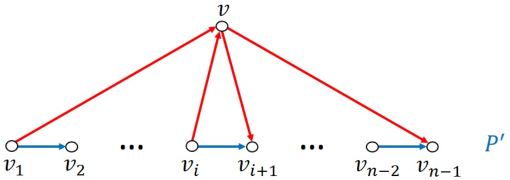

Theorem 2.3 (continued 2)

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Proof (continued). Then the path $(v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_{n-1})$ is a directed Hamilton path of T.

Since T is a tournament, at least one of (1), (2), or (3) must hold and so a tournament on n vertices has a Hamilton path. Therefore, by mathematical induction, the result holds for all $n \in \mathbb{N}$ and every tournament has a Hamilton path, as claimed. □

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Theorem 2.4

Theorem 2.4. Every loopless graph G contains a spanning bipartite subgraph F such that $d_F(v) \geq \frac{1}{2}d_G(v)$ for all $v \in V$.

Proof. Let G be a loopless graph. The empty spanning subgraph (i.e., the subgraph with no edges) is a spanning bipartite subgraph of G. Let $F = F[X, Y]$ be a spanning bipartite subgraph of G with the greatest possible number of edges (since G is finite, such a subgraph exists).

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Proof (continued). Symbolically, $E(F') = (E(F) \setminus \{e \in E(F) \mid e \text{ is incident to } v\}) \cup \{e \in E(G) \mid e \text{ is incident to } v \text{ and } e \notin E(F)\}$. Now, $d_F(v) = |\{e \in E(F) \mid e \text{ is incident to } v\}|$ and

$$d_G(v) = |\{e \in E(F) \mid e \text{ is incident to } v\}|$$
$$+|\{e \in E(G) \mid e \text{ is incident to } v \text{ and } e \notin E(F)\}|,$$

so $|\{e \in E(G) | e \text{ is incident to } v \text{ and } e \notin E(F)\}| = d_G(v) - d_F(v)$. Hence

$$e(F') = |E(F')| = (e(F) - d_F(v)) + (d_G(v) - d_F(v))$$
$$= e(F) + (d_G(v) - 2d_F(v)) > e(F)$$

where the inequality holds because we assumed $d_F(v) < \frac{1}{2}d_G(v)$. But this is a CONTRADICTION to the fact that $F = F[X, Y]$ is a spanning bipartite subgraph of G with the greatest possible number of edges. So the assumption that $d_F(v) < \frac{1}{2}d_G(v)$ for some $v \in V(F[X, Y])$ is false and it must be that $d_F(v) \geq \frac{1}{2}d_G(v)$ for all vertices v in $F = F[X, Y]$, as claimed. □

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Theorem 2.5

Theorem 2.5. Every graph with average degree at least $2k$, where $k \in \mathbb{N}$, has an induced subgraph with minimum degree at least $k+1$.

Proof. Let G be a graph with average degree $d(G) \geq 2k$. Let H be an induced subgraph of G with the largest possible average degree (which exists since G is a finite graph; and there are only finitely many such H since G only has finitely many induced subgraphs). Among such induced subgraphs H, choose one with the smallest number of vertices (which exists since there are only finitely many such H) and denote it as F.

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Theorem 2.5 (continued 1)

Proof (continued). ASSUME $d_F(v) \leq k$ for some vertex v of F. Consider the vertex-deleted subgraph $F' = F - v$. Note that F' is also an induced subgraph of G and $v(F') < v(F)$. So by the choice of F

$$d(F) > d(F'). \qquad (*)$$

Moreover

$$\begin{aligned} d(F') &= \frac{\sum_{v \in V(F')} d(v)}{|V(F')|} = \frac{2e(F')}{v(F')} \text{ by Theorem 1.1} \\ &= \frac{2e(F')}{v(F)-1} \geq \frac{2(e(F)-k)}{v(F)-1} \text{ since } d_F(v) \leq k \\ &\geq \frac{2e(F)-d(G)}{v(F)-1} \text{ since } d(G) \geq 2k \text{ by hypothesis} \\ &\geq \frac{2e(F)-d(F)}{v(F)-1} \text{ since } d(F) \geq d(G) \text{ as argued above} \ldots \end{aligned}$$

Theorem 2.5 (continued 2)

Proof (continued). ...

$$\begin{aligned} d(F') &\geq \frac{2e(F) - d(F)}{v(F) - 1} \text{ since } d(F) \geq d(G) \text{ as argued above} \\ &= \frac{\sum_{v \in V(F)} d(v) - \sum_{v \in V(F)} d(v)/v(F)}{v(F) - 1} \text{ by Theorem 1.1} \\ &= \frac{v(F) \sum_{v \in V(F)} d(v) - \sum_{v \in V(F)} d(v)}{v(F)(v(F) - 1)} \\ &= \frac{(v(F) - 1) \sum_{v \in V(F)} d(v)}{v(F)(v(F) - 1)} = \frac{\sum_{v \in V(F)} d(v)}{v(F)} = d(F). \end{aligned}$$

But we know $d(F) > d(F')$ from $(*)$ (and the choice of F), a CONTRADICTION. So the assumption that $d_F(v) \leq k$ for some vertex v of F is false and hence F is an induced subgraph of G such that $d_F(v) > k$ (or equivalently $d_F(v) \geq k + 1$) for all vertices v in F, as claimed. □

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&= \frac{\sum_{v \in V(F)} d(v) - \sum_{v \in V(F)} d(v)/v(F)}{v(F) - 1} \text{ by Theorem 1.1} \\
&= \frac{v(F) \sum_{v \in V(F)} d(v) - \sum_{v \in V(F)} d(v)}{v(F)(v(F) - 1)} \\
&= \frac{(v(F) - 1) \sum_{v \in V(F)} d(v)}{v(F)(v(F) - 1)} = \frac{\sum_{v \in V(F)} d(v)}{v(F)} = d(F).
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