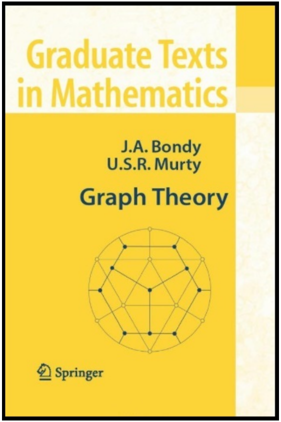
Graph Theory

Chapter 2. Subgraphs

2.4. Decompositions and Coverings—Proofs of Theorems



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Theorem 2.8

Theorem 2.8. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a decomposition of K_n into complete bipartite graphs. Then $k \geq n - 1$.

Proof. Let $V = V(K_n)$ and let complete bipartite graph F_i have bipartition (X_i, Y_i) for $1 \leq i \leq k$. Introduce n variables indexed by the vertices $v \in V$, denoted x_v. Consider the following system of $k+1$ homogeneous linear equations in the n variables x_v where $v \in V$:

$$\sum_{v \in V} x_v = 0, \text{ and } \sum_{v \in X_i} x_v = 0 \text{ for } 1 \leq i \leq k.$$

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ASSUME $k < n - 1$. Then this system consists of fewer than n equations (namely, $k + 1 < n$ equations) in n variables. Then by the "Fewer Equations than Unknowns" theorem (see my online notes for Linear Algebra [MATH 2010] on 1.6. Homogeneous Systems, Subspaces, and Bases; see Corollaries 1 and 2), the system has a nontrivial solution, $x_v = c_v$ where $v \in V$ and $c_v \neq 0$ for at least one $v \in V$.

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Proof (continued). Then

$$\sum_{v \in V} c_v = 0 \text{ and } \sum_{v \in X_i} c_v = 0 \text{ for } 1 \le i \le k. \qquad (*)$$

Next, for each edge $vw \in E(K_n)$, we consider the number $c_v c_w$. We sum these numbers in two ways. First, simply sum as $\sum_{vw \in E(K_n)} c_v c_w$ and second sum using the fact that $\mathcal{F}$ is a decomposition of K_n. In the second sum, for $F_i \in \mathcal{F}$ with bipartition (X_i, Y_i) we have all edges of the form vw where $v \in X$ and $w \in Y$ so that

$$\sum_{vw \in E(F_i)} c_v c_w = \left(\sum_{v \in X_i} c_v \right) \left(\sum_{w \in Y_i} c_w \right);$$

then the sum over all edges in the F_is (and so all edges of K_n) is

$$\sum_{i=1}^{k} \left(\sum_{v \in X_i} c_v \right) \left(\sum_{w \in Y_i} c_w \right).$$

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Proof (continued). So we must have

$$\sum_{v,w \in V,\, v \neq w} c_v c_w = \sum_{vw \in E(K_n)} c_v c_w = \sum_{i=1}^{k} \left(\sum_{v \in X_i} c_v \right) \left(\sum_{w \in Y_i} c_w \right). \quad (**)$$

Therefore

$$\begin{aligned} 0 &= \left(\sum_{v \in V} c_v \right)^2 \text{ by } (*) \\ &= \left(\sum_{v \in V} c_v \right) \left(\sum_{v \in V} c_v \right) \\ &= \sum_{v \in V} c_v^2 + 2 \sum_{v,w \in V,\, v \neq w} c_v c_w \text{ because for real numbers } x_i, \\ & \qquad 1 \leq i \leq m \text{ we have } \left(\sum_{i=1}^{m} x_i \right)^2 = \sum_{i=1}^{m} x_i^2 + 2 \sum_{i=1, j=1, i \neq j}^{m} x_i x_j \end{aligned}$$

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$$\begin{aligned} 0 &= \sum_{v \in V} c_v^2 + 2 \sum_{v,w \in V,\, v \neq w} c_v c_w \\ &= \sum_{v \in V} c_v^2 + 2 \sum_{i=1}^{k} \left(\sum_{v \in X_i} c_v \right) \left(\sum_{v \in Y_i} c_w \right) \text{ by } (**) \\ &= \sum_{v \in V} c_v^2 \text{ since } \sum_{v \in X_i} c_v = 0 \text{ for } 1 \leq i \leq k \text{ by } (*) \\ &> 0 \text{ since some } c_v \neq 0, \end{aligned}$$

a CONTRADICTION. So the assumption that $k < n - 1$ is false and we must have $k \geq n - 1$, as claimed. □