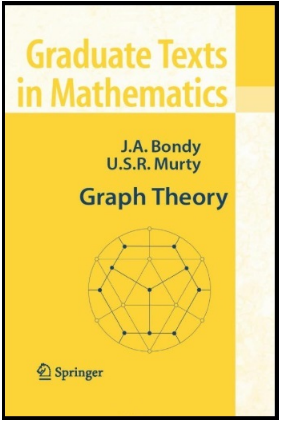
Graph Theory

Chapter 2. Subgraphs

2.5. Edge Cuts and Bonds—Proofs of Theorems



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Theorem 2.10

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Suppose G is even. Then (by definition of "even") $d(v)$ is even for each $v \in V$, and so $\sum_{v \in V} d(v)$ is even. So by Theorem 2.9, it follows that $|\partial(X)|$ must be even for every $X \subset V$. □

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Proposition 2.11

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Proof. Bondy and Murty give crude Venn diagrams to explain this proof. We also give a "proof by picture," but we use color coded edges to describe edge cuts. First, for given $X, Y \subset V$ we partition V as

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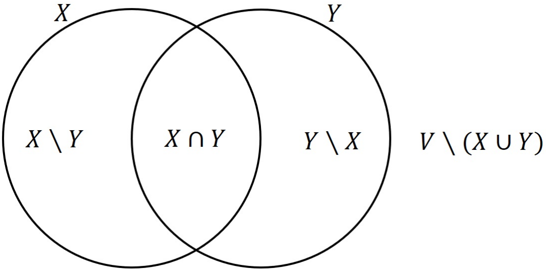

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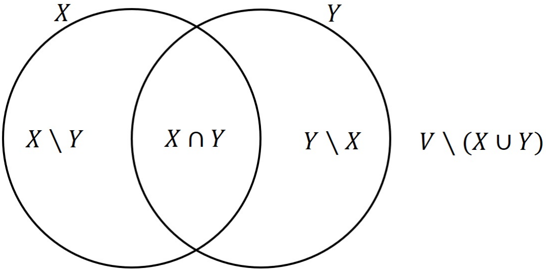
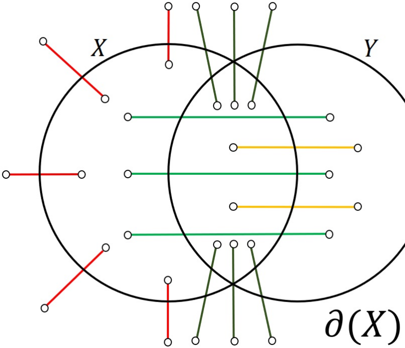


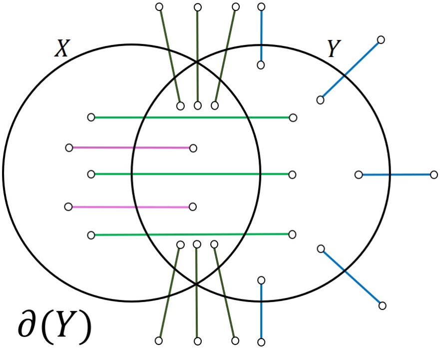
Proposition 2.11 (continued 1)

Proof (continued). Based on these four disjoint sets, we now describe $\partial(X)$, $\partial(Y)$, and $\partial(X \triangle Y)$. The edges in $\partial(X)$ are of the following four types: (1) edges with one end in $X \setminus Y$ and one end in $V \setminus (X \cup Y)$ (in red), (2) edges with one end in $X \setminus Y$ and one end in $Y \setminus X$ (in green), (3) edges with one end in $X \cap Y$ and one end in $V \setminus (X \cup Y)$ (in brown), and (4) edges with one end in $X \cap Y$ and one end in $Y \setminus X$ (in orange).



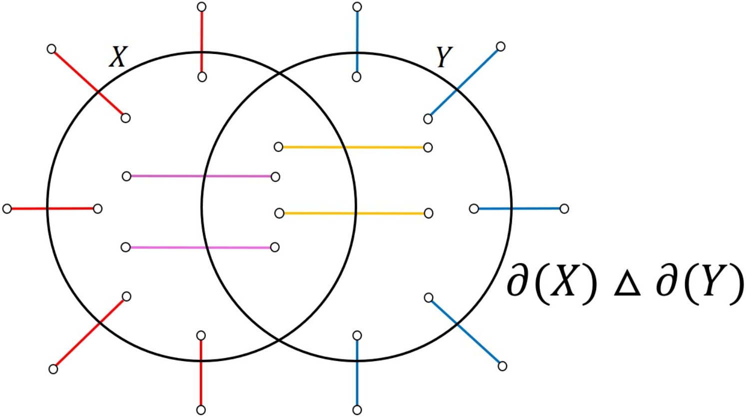
Proposition 2.11 (continued 2)

Proof (continued). The edges in $\partial(Y)$ are of the following four types: (5) edges with one end in $Y \setminus X$ and one end in $V \setminus (X \cup Y)$ (in blue), (6) edges with one end in $Y \setminus X$ and one end in $X \setminus Y$ (in green), (7) edges with one end in $X \cap Y$ and one end in $V \setminus (X \cup Y)$ (in brown), and (8) edges with one end in $X \cap Y$ and one end in $X \setminus Y$ (in pink).



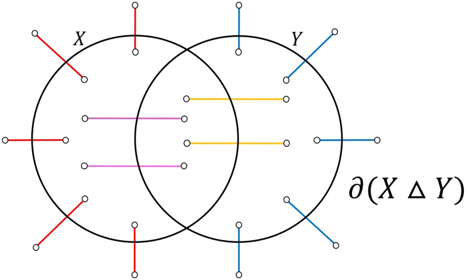
Proposition 2.11 (continued 3)

Proof (continued). Now $\partial(X) \triangle \partial(Y) = (\partial(X) \setminus \partial(Y)) \cup (\partial(Y) \setminus \partial(X))$ consists of the edges colored red, orange, blue, and pink.



Proposition 2.11 (continued 4)

Proof (continued). Next, $X \triangle Y = (X \setminus Y) \dot\cup (Y \setminus X)$ and the edges of $\partial(X \triangle Y)$ are of the following four types: (9) edges with one end in $X \setminus Y$ and one end in $V \setminus (X \cup Y)$ (in red), (10) edges with one end in $X \setminus Y$ and one end in $X \cap Y$ (in pink), (11) edges with one end in $Y \setminus X$ and one end in $V \setminus (X \cup Y)$ (in blue), and (12) edges with one end in $Y \setminus X$ and one end in $X \cap Y$ (in orange).



So $\partial(X \triangle Y)$ consists of the edges colored red, orange, blue, and pink. Therefore, $\partial(X) \triangle \partial(Y) = \partial(X \triangle Y)$, as claimed. □

Theorem 2.15

Theorem 2.15. In a connected graph G, a nonempty edge cut $\partial(X)$ is a bond if and only if both $G[X]$ and $G[V \setminus X]$ are connected.

Proof. First, suppose $\partial(X)$ is a bond. Let Y be a nonempty proper subset of X. Since G is connected by hypothesis, then both $\partial(Y)$ and $\partial(X \setminus Y)$ are nonempty (or else Y and $V \setminus Y$, or $X \setminus Y$ and $V \setminus (X \setminus Y)$ form a "separation" of G, contradicting the fact that G is connected).

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Proof (continued). Second, suppose $\partial(X)$ is not a bond. Then $\partial(X)$ is not a minimal edge cut. So there is some subset Y of V with $\varnothing \neq \partial(Y) \subsetneq \partial(X)$, $Y \neq \varnothing$ (since $\partial(\varnothing) = \varnothing$), and $Y \neq V$ (since $\partial(V) = \varnothing$).

We claim, without loss of generality, that $X \cap Y \neq \varnothing$. Suppose, to the contrary, that $X \cap Y = \varnothing$. Then if $e \in \partial(Y) \subseteq \partial(X)$ then one end of e is in Y and one end is in $V \setminus Y$, AND one end of e is in X and one end is in $V \setminus X$. In the event that $X \cap Y = \varnothing$ then e cannot have one end in X and one end in $V \setminus (X \cup Y)$ (since then $e \notin \partial(Y)$) nor can e have one end in Y and one end in $V \setminus (X \cup Y)$ (since then $e \notin \partial(X)$). So for $e \in \partial(Y)$, e has one end in X and one end in Y; in fact, $\partial(Y) = E[X, Y]$.

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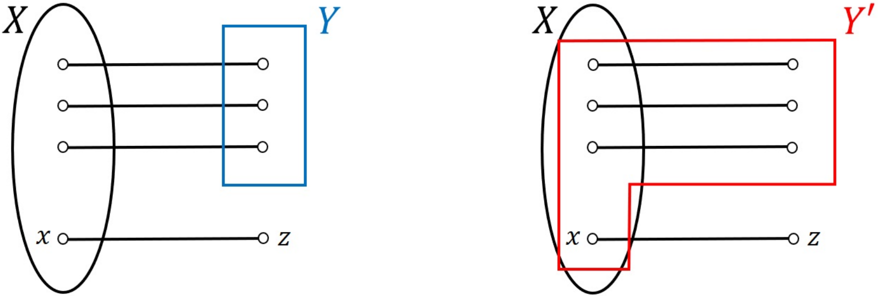
Theorem 2.15 (continued 2)

Proof (continued). Let $Y' = X \cup Y$. Then $\partial(Y') = \partial(X) \setminus E[X, Y] \subsetneq \partial(X)$ (since $E[X, Y] = \partial(Y) \neq \varnothing$ here). Since edge $\{x, z\} \in \partial(X \cup Y) = \partial(Y')$ then $Y' \neq V$. So we can replace Y with Y' where $X \cap Y' \neq \varnothing$. So we may assume without loss of generality that $X \cap Y \neq \varnothing$ above, as claimed.



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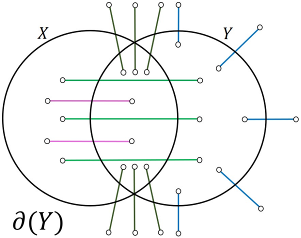
Proof (continued). Since $\partial(Y) \subset \partial(X)$ then we must have $E[X \cap Y, X \setminus Y] = \varnothing$ (in pink) and $E[Y \setminus X, V \setminus (X \cup Y)] = \varnothing$ (in blue).



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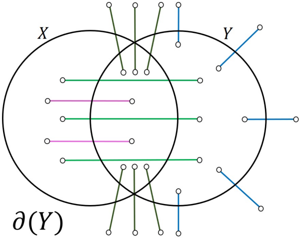

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