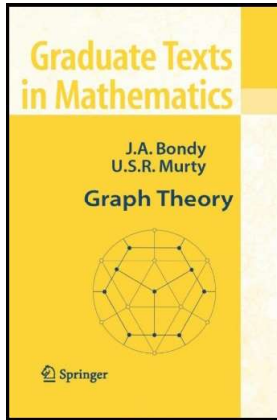


# Graph Theory

## Chapter 2. Subgraphs

### 2.6. Even Subgraphs—Proofs of Theorems



## Proposition 2.18

**Proposition 2.18.** In any graph, every (edge set of an) even subgraph meets every edge cut in an even number of edges.

**Proof.** Let  $C$  be a cycle and let  $\partial(X)$  be an edge cut. Each vertex of  $C$  is either in  $X$  or in  $V \setminus X$ . As  $C$  is “traversed,” the number of times it crosses from  $X$  to  $V \setminus X$  must be the same as the number of times it crosses from  $V \setminus X$  to  $X$  (since it “begins” and “ends” at the same vertex). Thus  $|E(C) \cap \partial(X)| = |E(C) \cap E[X, V \setminus X]|$  is even. So the claim holds for even subgraphs which are cycles.

By Theorem 2.17, every even subgraph is a (edge-) disjoint union of cycles, say  $C_1, C_2, \dots, C_k$ . Since each such  $C_i$  intersects  $\partial(X)$  in an even number of edges as shown above, then  $\cup_{i=1}^k C_i$  intersects  $\partial(X)$  in an even number of edges and so the result holds for all even subgraphs, as claimed.  $\square$

## Corollary 2.16

**Corollary 2.16.** The symmetric difference of two even subgraphs is an even subgraph.

**Proof.** Let  $F_1$  and  $F_2$  be even subgraphs of a graph  $G$ , and let  $X$  be a subset of  $V$ . By Proposition 2.13,

$$\partial_{F_1 \Delta F_2}(X) = \partial_{F_1}(X) \Delta \partial_{F_2}(X). \quad (*)$$

By Theorem 2.10, both  $|\partial_{F_1}(X)|$  and  $|\partial_{F_2}(X)|$  are even since  $F_1$  and  $F_2$  are even. So  $|\partial_{F_1}(X) \setminus \partial_{F_2}(X)| = |\partial_{F_1}(X) \setminus (\partial_{F_1}(X) \cap \partial_{F_2}(X))|$  and  $|\partial_{F_2}(X) \setminus \partial_{F_1}(X)| = |\partial_{F_2}(X) \setminus (\partial_{F_1}(X) \cap \partial_{F_2}(X))|$  are of the same parity; both are of the parity of  $|\partial_{F_1}(X) \cap \partial_{F_2}(X)|$ . Hence

$$|\partial_{F_1}(X) \Delta \partial_{F_2}(X)| = |(\partial_{F_1}(X) \setminus \partial_{F_2}(X)) \cup (\partial_{F_2}(X) \setminus \partial_{F_1}(X))|$$

is even. So from (\*),  $|\partial_{F_1 \Delta F_2}(X)|$  is even. Since  $X$  is an arbitrary subset of  $V$ , then, again by Theorem 2.10,  $F_1 \Delta F_2$  is even, as claimed.  $\square$

## Theorem 2.6.A

**Theorem 2.6.A.** Let  $G$  be a graph. The set of all even subgraphs of  $G$  form a subspace of the edge space of  $G$ .

**Proof.** Recall that a set of vectors from a vector space form a subspace if the set is closed under linear combinations (see Theorem 3.2, “Test for Subspace,” in my online notes for Linear Algebra (MATH 2010) on [3.2. Basic Concepts of Vector Spaces](#)). Let  $W \subset \mathcal{E}(G)$  be the set of edge sets of even subgraphs of  $G$ . Let  $E_1, E_2 \in W$  and let  $a, b \in GF(2)$ . We consider  $aE_1 + bE_2$ . If  $a = b = 0$  then  $aE_1 + bE_2 = \emptyset \Delta \emptyset = \emptyset$  and since  $|\emptyset| = 0$  is even then  $aE_1 + bE_2 \in W$ . If  $a = 1$  and  $b = 0$  then  $aE_1 + bE_2 = E_1 \Delta \emptyset = E_1 \in W$ . If  $a = 0$  and  $b = 1$  then  $aE_1 + bE_2 = \emptyset \Delta E_2 = E_2 \in W$ . If  $a = b = 1$  then  $aE_1 + bE_2 = E_1 \Delta E_2$  and by Corollary 2.16  $E_1 \Delta E_2$  is an even subgraph of  $G$  since  $E_1$  and  $E_2$  are even subgraphs of  $G$ , so that  $aE_1 + bE_2 \in W$ . Therefore  $W$  is closed under linear combinations and hence is a subspace of  $\mathcal{E}(G)$ , as claimed.  $\square$