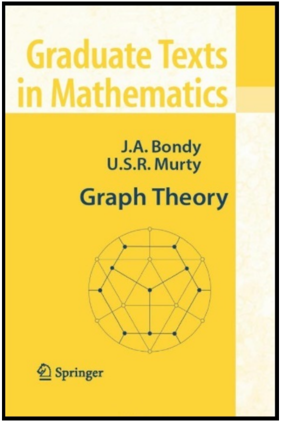
Graph Theory

Chapter 2. Subgraphs

2.6. Even Subgraphs—Proofs of Theorems



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Proof. Let F_1 and F_2 be even subgraphs of a graph G, and let X be a subset of V. By Proposition 2.13,

$$\partial_{F_1 \triangle F_2}(X) = \partial_{F_1}(X) \triangle \partial_{F_2}(X). \qquad (*)$$

By Theorem 2.10, both $|\partial_{F_1}(X)|$ and $|\partial_{F_2}(X)|$ are even since F_1 and F_2 are even. So $|\partial_{F_1}(X) \setminus \partial_{F_2}(X)| = |\partial_{F_1}(X) \setminus (\partial_{F_1}(X) \cap \partial_{F_2}(X))|$ and $|\partial_{F_2}(X) \setminus \partial_{F_1}(X)| = |\partial_{F_2}(X) \setminus (\partial_{F_1}(X) \cap \partial_{F_2}(X))|$ are of the same parity; both are of the parity of $|\partial_{F_1}(X) \cap \partial_{F_2}(X)|$.

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is even. So from $(*)$, $|\partial_{F_1 \triangle F_2}(X)|$ is even. Since X is an arbitrary subset of V, then, again by Theorem 2.10, $F_1 \triangle F_2$ is even, as claimed. □

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Proposition 2.18

Proposition 2.18. In any graph, every (edge set of an) even subgraph meets every edge cut in an even number of edges.

Proof. Let C be a cycle and let $\partial(X)$ be an edge cut. Each vertex of C is either in X or in $V \setminus X$. As C is "traversed," the number of times it crosses from X to $V \setminus X$ must be the same as the number of times it crosses from $V \setminus X$ to X (since it "begins" and "ends" at the same vertex). Thus $|E(C) \cap \partial(X)| = |E(C) \cap E[X, V \setminus X]|$ is even. So the claim holds for even subgraphs which are cycles.

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By Theorem 2.17, every even subgraph is a (edge-) disjoint union of cycles, say $C_1, C_2, \ldots, C_k$. Since each such C_i intersects $\partial(X)$ in an even number of edges as shown above, then $\cup_{i=1}^{k} C_i$ intersects $\partial(X)$ in an even number of edges and so the result holds for all even subgraphs, as claimed. □

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Proof. Recall that a set of vectors from a vector space form a subspace if the set is closed under linear combinations (see Theorem 3.2, "Test for Subspace," in my online notes for Linear Algebra (MATH 2010) on 3.2. Basic Concepts of Vector Spaces). Let $W \subset \mathcal{E}(G)$ be the set of edge sets of even subgraphs of G. Let $E_1, E_2 \in W$ and let $a, b \in GF(2)$. We consider $aE_1 + bE_2$.

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