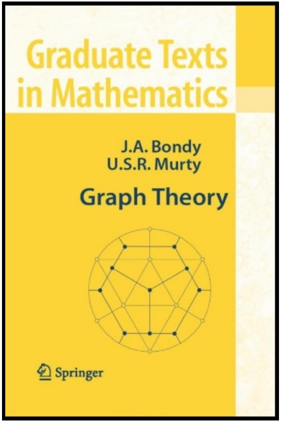
Graph Theory

Chapter 2. Subgraphs

2.7. Graph Reconstruction—Proofs of Theorems



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Lemma 2.20. KELLY'S LEMMA. For any two graphs F and G such that $v(F) < v(G)$, the parameter $\binom{G}{F}$ is a reconstructible parameter.

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Corollary 2.21

Corollary 2.21. For any two graphs F and G, the number of subgraphs of G that are isomorphic to F and includes a given vertex v is a reconstructible parameter.

Proof. The number of subgraphs of G that are isomorphic to F is $\binom{G}{F}$. The number of these that exclude vertex v is $\binom{G-v}{F}$. So the number of subgraphs of G that are isomorphic to F and include vertex v is $\binom{G}{F} - \binom{G-v}{F}$. Since $\binom{G}{F}$ is a reconstructible parameter by Kelly's Lemma (Lemma 2.20) and $\binom{G-v}{F}$ is a parameter of $G-v$, then $\binom{G}{F} - \binom{G-v}{F}$ is a reconstructible parameter, as claimed. □

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Corollary 2.22

Corollary 2.22. The size and the degree sequence are reconstructible parameters.

Proof. The size of graph G is $m = e(G) = \binom{G}{K_2}$ and so is a constructible parameter by Kelly's Lemma (Lemma 2.20). With $F = K_2$, Corollary 2.21 implies that the number of edges in G that include a vertex v is a constructible parameter. But this parameter is just the degree of v. So the degree of each vertex of G (and hence the degree sequence of G) is a constructible parameter, as claimed. □

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Theorem 2.25

Theorem 2.25. THE MÖBIUS INVERSION FORMULA.
Let $f : 2^T \to \mathbb{R}$ (here, 2^T represents the power set of T, $2^T = \mathcal{P}(T)$) be a real-valued function defined on the subsets of a finite set T. Define the function $g : 2^T \to \mathbb{R}$ by $g(S) = \sum_{S \subseteq X \subseteq T} f(X)$. Then for all $S \subseteq T$,

$$f(S) = \sum_{S \subseteq X \subseteq T} (-1)^{|X|-|S|} g(X).$$

Proof. First, by the Binomial Theorem, for $n \in \mathbb{N}$ we have

$$0 = (1 + (-1))^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$= \sum_{0 \le k \le n} \binom{n}{k} (-1)^k = \sum_{s \le k \le n+s} \binom{n}{k-s} (-1)^{k-s}$$

where $s \in \mathbb{N} \cup \{0\}$ is any constant.

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Theorem 2.25 (continued 1)

Proof (continued). Now for given set Y and $S \subseteq Y$ there are $\binom{|Y|-|S|}{|X|-|S|}$ sets X with $|X| = k$ and $S \subseteq X \subseteq Y$ (since we create such a set X by choosing $|X| - |S|$ elements of set $Y \setminus S$ and then X consists of these elements along with the elements of set S). So for given finite sets S and Y where $S \subseteq Y$, $|S| = s$, and $n = |Y| - |S| \neq \varnothing$ (so that $S \neq Y$) we have

$$\begin{aligned} 0 &= \sum_{s \le k \le n+s} \binom{n}{k-s} (-1)^{k-2} \\ &= \sum_{s \le |X| \le n+s} (-1)^{|X|-s} \text{ where } X \text{ ranges over the } \binom{|Y|-s}{|X|-s} \text{ sets} \\ &\qquad \text{that are supersets of } S \text{ and subsets of } Y \\ &= \sum_{|S| \le |X| \le |Y|} (-1)^{|X|-|S|} \text{ where } X \text{ is as above.} \qquad (*) \end{aligned}$$

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Proof (continued). Now $(*)$ holds for given finite set Y and any $S \subsetneq Y$. With set Y fixed, S ranging over all subsets of Y (except $S = Y$), and X ranging over all sets that ae supersets of given set S and subsets of Y, we have from $(*)$ that

$$0 = \sum_{S \subset \neq Y} \left(\sum_{S \subseteq X \subseteq Y} (-1)^{|X|-|S|} \right)$$
$$= \sum_{S \subseteq X \subseteq Y,\, S \neq Y} (-1)^{|X|-|S|} \text{ where set } Y \text{ is fixed and sets } S$$

and X range over all sets satisfying $S \subseteq X \subseteq Y$ and $X \neq Y$.

Of course if $S = Y$ then $S \subseteq X \subseteq Y$ implies $S = X = Y$ and $\sum_{S \subseteq X \subseteq Y} (-1)^{|X|-|S|} = 1$. So for any finite set Y, we have

$$\sum_{X \subseteq X \subseteq Y} (-1)^{|X|-|S|} = \begin{cases} 0 & \text{if } S \neq Y \\ 1 & \text{if } S = Y. \end{cases}$$

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Proof (continued). Therefore

$$
\begin{aligned}
f(S) &= \sum_{S\subseteq Y\subseteq T} f(Y)\left(\sum_{S\subseteq X\subseteq Y}(-1)^{|X|-|S|}\right) \text{ since the quantity} \\
& \quad \text{in parentheses is 0 except when } Y=S \\
&= \sum_{S\subseteq Y\subseteq T}\sum_{S\subseteq X\subseteq Y} f(Y)(-1)^{|X|-|S|} = \sum_{S\subseteq X\subseteq Y\subseteq T} f(Y)(-1)^{|X|-|S|} \\
&= \sum_{S\subseteq X\subseteq T}\sum_{X\subseteq Y\subseteq T}(-1)^{|X|-|S|} f(Y) \\
&= \sum_{S\subseteq X\subseteq T}(-1)^{|X|-|S|}\left(\sum_{X\subseteq Y\subseteq T} f(Y)\right) \\
&= \sum_{S\subseteq X\subseteq T}(-1)^{|X|-|S|} g(X) \text{ by the definition of } g,
\end{aligned}
$$

as claimed. □

Lemma 2.26

Lemma 2.26. Nash-Williams' Lemma.
Let G be a graph, F a spanning subgraph of G, and H an edge reconstruction of G that is not isomorphic to G. Then

$$|G \to G|_F - |G \to H|_F = (-1)^{e(G)-e(F)}\mathrm{aut}(G).$$

Proof. Since F is a subgraph of G, by (2.6) and (2.7) we have $\sum_{F \subseteq X \subseteq G} |G \to H|_X = \mathrm{aut}(G)\binom{H}{F}$. Now define $f(X) = |G \to H|_X$ so that f maps $2^{E(G)}$ (the power set of the edge set of G) into $\mathbb{R}$.

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$$\begin{aligned} g(F) &= \sum_{F\subseteq X\subseteq G} f(X) = \sum_{F\subseteq X\subseteq G} |G \to H|_X \\ &= |F \to H| \text{ by (2.6).} \end{aligned}$$

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Lemma 2.26 (continued 1)

Proof (continued). Then by the Möbius Inversion Formula (Theorem 2.25)

$$\begin{aligned} f(F) &= \sum_{F\subseteq X\subseteq G} (-1)^{|X|-|F|} g(X) = \sum_{F\subseteq X\subseteq G} (-1)^{e(X)-e(F)} |X \to H| \\ &= \sum_{F\subseteq X\subseteq G} (-1)^{e(X)-e(F)} \operatorname{aut}(X) \binom{H}{X} \text{ by (2.7).} \end{aligned}$$

By the definition of f, $f(F) = |G \to H|_F$, so we have

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Therefore,

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Lemma 2.26 (continued 2)

Proof (continued). Now H is an edge reconstruction of G by hypothesis so for each $X \subsetneq G$ (where $e(X) < e(G)$) we have by Kelly's Lemma Edge Version (Lemma 2.24) that $\binom{G}{X} = \binom{H}{X}$. So we then have

$$|G \to G|_F - |G \to H|_F = (-1)^{e(G)-e(F)}\text{aut}\left(\binom{G}{G} - \binom{H}{G}\right)$$

$$= (-1)^{e(G)-e(F)}\text{aut}(G)$$

where the last equality holds because $\binom{G}{G} = 1$, and $\binom{H}{G} = 0$ since $e(H) = e(G)$ but $h \not\cong G$ by hypothesis. So the equation holds, as claimed. □

Theorem 2.27

Theorem 2.27. A graph G is edge reconstructible if there exists a spanning subgraph F of G such that either of the following two conditions holds:

(i) $|G \to H|_F$ takes the same value for all edge reconstructions H of G.

(ii) $|F \to G| < 2^{e(G)-e(F)-1}\mathrm{aut}(G)$.

Proof. Let H be an edge reconstruction of G. We show that each of the two given conditions contradict Nash-Williams' Lemma so that we can conclude that $H \cong G$ and hence G is reconstructible.

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Suppose condition (i) holds and ASSUME $H \not\cong G$. Then the Nash Williams' Lemma, which gives

$$|G \to G|_F - |G \to H|_F - (-1)^{e(G)-e(F)}\text{aut}(G),$$

implies $|G \to G|_F - |G - H|_F = 0$ (since G is a reconstruction of G), but $(-1)^{e(G)-e(F)}\text{aut}(G) \neq 0$, a CONTRADICTION.

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Suppose condition (i) holds and ASSUME $H \ncong G$. Then the Nash Williams' Lemma, which gives

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Proof (continued). So the assumption that $H \ncong G$ is false and we must have $H \cong G$. That is, G is reconstructible, as claimed.

Suppose condition (ii) holds and ASSUME $H \ncong G$. Then

$$\sum_{F \subseteq X \subseteq G} |G \to G|_X = |F \to G| \text{ by (2.6)}$$

$$< 2^{e(G)-e(F)-1}\text{aut}(G) \text{ by condition (ii).} \quad (*)$$

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We now count the number of graphs X which satisfy $F \subseteq X \subseteq G$. Such a graph X must contain all edges of F and none, some, or all of the edges in $E(G) \setminus E(F)$. Since $|E(G) \setminus E(F)| = e(G) - e(F)$ then there are $2^{e(G)-e(F)}$ possible graphs X. Index these $2^{e(G)-e(F)}$ graphs as X_i for $i = 1, 2, \ldots, 2^{e(G)-e(F)}$. Defining $x_i = |G \to G|_{X_i}$, $(*)$ implies

$$\sum_{i=1}^{2^{e(G)-e(F)}} |G \to G|_{X_i} = \sum_{i=1}^{2^{e(G)-e(f)}} x_i < e^{e(G)-e(F)-1}\text{aut}(G). \quad (**)$$

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Suppose condition (ii) holds and ASSUME $H \not\cong G$. Then

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Proof (continued). ASSUME half or more of the x_i satisfy $x_i \geq \mathrm{aut}(G)$. Then

$$\sum_{i=1}^{2^{e(G)-e(F)}} x_i \geq 2^{e(G)-e(F)-1}\mathrm{aut}(G),$$

CONTRADICTING $(**)$. So we must have less than half of the x_i satisfying $x_i \geq \mathrm{aut}(G)$ and hence we must have more than half of the x_i satisfying $x_i < \mathrm{aut}(G)$. Now $e(G) - e(F)$ (when $X_i = F$) where at least half of the values are even (exactly half when $e(G) - e(F)$ is odd and more than half when $e(G) - e(F)$ is even). Since more than half of the x_i satisfy $x_i < \mathrm{aut}(G)$, there must be some index i^* where $x_{i^*} = |G \to G|_{X_{i^*}} < \mathrm{aut}(G)$ and $e(G) - e(X_{i^*})$ is even. Denote this X_{i^*} as X.

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Proof (continued). Then with $F = X$ in Nash-Williams' Lemma (Lemma 2.26; we have assumed $H \not\cong G$) we have

$$\begin{aligned} |G \to G|_X - |G \to H|_X &= (-1)^{e(G)-e(F)}\text{aut}(G) \\ &= \text{aut}(G) \text{ since } e(G) - e(X) \text{ is even} \\ &< \text{aut}(G) - |G \to H|_X \text{ since} \\ &\quad\ |G \to G|_X < \text{aut}(G). \end{aligned}$$

But this implies that $0 < -|G \to H|_X$, a CONTRADICTION since $|G \to H|_X \geq 0$. So the assumption that $H \not\cong G$ is false and we must have $H \cong G$. That is, G is reconstructible, as claimed. □

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