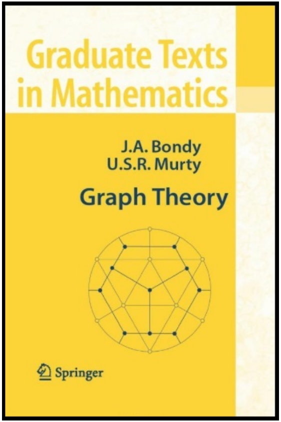
Graph Theory

Chapter 3. Connected Graphs

3.1. Walks and Connection—Proofs of Theorems



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Lemma 3.1.A

Lemma 3.1.A. Let $\mathbf{J}$ be the $n \times n$ matrix with all entries 1. Then the eigenvalues of $\mathbf{J}$ are 0 (with algebraic multiplicity $n-1$) and n (with algebraic multiplicity 1).

Proof. We show that the characteristic polynomial is $(-1)^n \lambda^{n-1}(\lambda - n)$, from which the result will follow. We establish this by mathematical induction. For $n = 1$, we have $|\mathbf{J} - \lambda \mathbf{I}| = 1 - \lambda = (-1)^1 \lambda^0 (\lambda - 1)$, establishing the base case. Notice that for $n = 2$,

$$|\mathbf{J} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1$$

$$= \lambda^2 - 2\lambda = \lambda(\lambda - 2) = (-1)^2 \lambda^1 (\lambda - 2)$$

establishing another case. Suppose the claim holds for $n = k - 1$ and consider the case $n = k$.

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establishing another case. Suppose the claim holds for $n = k - 1$ and consider the case $n = k$.

Lemma 3.1.A (continued 1)

Proof (continued). Then expanding determinants along the first column,

$$|\mathbf{J} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ 1 & 1 & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1\cdots & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 & \cdots & 1 \\ 1 & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\lambda \end{vmatrix}$$

Lemma 3.1.A (continued 2)

Proof (continued).

$$-(1)\begin{vmatrix} \mathbf{1} & 1 & 1 & \cdots & 1 \\ \mathbf{1} & 1-\lambda & 1 & \cdots & 1 \\ \mathbf{1} & 1 & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} & 1 & 1 & \cdots & 1-\lambda \end{vmatrix}$$

$$+(1)\begin{vmatrix} 1 & \mathbf{1} & 1 & \cdots & 1 \\ 1-\lambda & \mathbf{1} & 1-\lambda & \cdots & 1 \\ 1 & \mathbf{1} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mathbf{1} & 1 & \cdots & 1-\lambda \end{vmatrix}$$

Lemma 3.1.A (continued 3)

Proof (continued).

$$-(1)\begin{vmatrix} 1 & 1 & \mathbf{1} & 1 & \cdots & 1 \\ 1-\lambda & 1 & \mathbf{1} & 1-\lambda & \cdots & 1 \\ 1 & 1-\lambda & \mathbf{1} & 1 & \cdots & 1 \\ 1 & 1 & \mathbf{1} & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \mathbf{1} & 1 & \cdots & 1-\lambda \end{vmatrix} + \cdots$$

$$+(-1)^{k-1}(1)\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & \mathbf{1} \\ 1-\lambda & 1 & 1 & \cdots & 1 & \mathbf{1} \\ 1 & 1-\lambda & 1 & \cdots & 1 & \mathbf{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1-\lambda & \mathbf{1} \end{vmatrix}. \qquad (*)$$

Lemma 3.1.A (continued 4)

Proof (continued). By the induction hypothesis, the first term in $(*)$ is $(1-\lambda)(-1)^{k-1}(\lambda^{k-2}(\lambda-(k-1))$. Now each of the other $k-1$ cofactors involve determinants of matrices with one column of all 1's (in bold faced; let this be column ℓ), to the left of which are all entries of 1 except for entries of $1-\lambda$ just below the diagonal entries, and to the right of which are all entries of 1 except for entries of $1-\lambda$ along the diagonal. First we use the first row of these matrices to eliminate the other entries of 1. We subtract row 1 from each of the rows below it producing a $(k-1)\times(k-1)$ matrix with first row all 1's, to the left of this column are all entries of 0 except for entries of $-\lambda$ just below the diagonal entries, and to the right of which are all entries of 0 except for entries of $-\lambda$ along the diagonal:

Lemma 3.1.A (continued 4)

Proof (continued). By the induction hypothesis, the first term in $(*)$ is $(1-\lambda)(-1)^{k-1}(\lambda^{k-2}(\lambda-(k-1))$. Now each of the other $k-1$ cofactors involve determinants of matrices with one column of all 1's (in bold faced; let this be column ℓ), to the left of which are all entries of 1 except for entries of $1-\lambda$ just below the diagonal entries, and to the right of which are all entries of 1 except for entries of $1-\lambda$ along the diagonal. First we use the first row of these matrices to eliminate the other entries of 1. We subtract row 1 from each of the rows below it producing a $(k-1)\times(k-1)$ matrix with first row all 1's, to the left of this column are all entries of 0 except for entries of $-\lambda$ just below the diagonal entries, and to the right of which are all entries of 0 except for entries of $-\lambda$ along the diagonal:

Lemma 3.1.A (continued 5)

Proof (continued).

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & \mathbf{1} & \cdots & 1 & 1 \\ 1-\lambda & 1 & 1 & \cdots & \mathbf{1} & \cdots & 1 & 1 \\ 1 & 1-\lambda & 1 & \cdots & \mathbf{1} & \cdots & 1 & 1 \\ 1 & 1 & 1-\lambda & \cdots & \mathbf{1} & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \mathbf{1} & \cdots & 1-\lambda & 1 \\ 1 & 1 & 1 & \cdots & \mathbf{1} & \cdots & 1 & 1-\lambda \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & \mathbf{1} & \cdots & 1 & 1 \\ -\lambda & 0 & 0 & \cdots & \mathbf{0} & \cdots & 0 & 0 \\ 0 & -\lambda & 0 & \cdots & \mathbf{0} & \cdots & 0 & 0 \\ 0 & 0 & -\lambda & \cdots & \mathbf{0} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{0} & \cdots & -\lambda & 0 \\ 0 & 0 & 0 & \cdots & \mathbf{0} & \cdots & 0 & -\lambda \end{bmatrix}$$

Lemma 3.1.A (continued 6)

Proof (continued). These new matrices have the same determinants as the original (respective) matrices (see Theorem 4.2.A. Properties of the Determinant from my online Linear Algebra [MATH 2010] notes on 4.2. The Determinant of a Square Matrix). Expanding the determinants of the remaining new matrices along the column consisting of a 1 followed by 0's (the ℓth column) gives

$$(-1)^{1+\ell}(1)\begin{vmatrix} -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda \end{vmatrix} = (-1)^{1+\ell}(1)(-1)^{k-2}\lambda^{k-2}.$$

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Therefore $|\mathbf{J} - \lambda \mathbf{I}|$

$$= (1-\lambda)(-1)^{k-1}(\lambda^{k-2}(\lambda-(k-1))) + \sum_{\ell=1}^{k-1}(-1)^{\ell}(1)(-1)^{1+\ell}(-1)^{k-2}\lambda^{k-2}$$

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Proof (continued).

$$(1-\lambda)(-1)^{k-1}(\lambda^{k-2}(\lambda-k+1))+(k-1)(-1)^{k-1}\lambda^{k-2}$$

$$=(-1)^{k-1}\lambda^{k-2}((1-\lambda)(\lambda-k+1)+k-1)$$

$$=(-1)^{k-1}\lambda^{k-2}(\lambda-k+1-\lambda^2+\lambda k-\lambda+k-1)$$

$$=(-1)^{k-1}\lambda^{k-2}(-\lambda^2+\lambda k)=(-1)^{k}\lambda^{k-2}(\lambda^2-\lambda k)=(-1)^{k}\lambda^{k-1}(\lambda-k).$$

So the result holds for $n=k$ and therefore by Mathematical Induction holds for all $n \in \mathbb{N}$. □

Theorem 3.1

Theorem 3.1. The Friendship Theorem. Let G be a simple graph in which any two vertices (people) have exactly one common neighbor (friend). Then G has a vertex of degree $n-1$ (everyone's friend).

Proof. We give a proof by contradiction. ASSUME G is a friendship graph and $\Delta < n-1$. We show first that G is regular.

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Consider two nonadjacent vertices x and y where, say, $d(x) \geq d(y)$. By hypothesis, x and y have exactly one common neighbor; denote it as z. For each neighbor v of x other than z, denote by $f(v)$ the common neighbor of v and y. Let $N(x) = \{z, v_1, v_2, \ldots, v_\ell\}$. Then we have:

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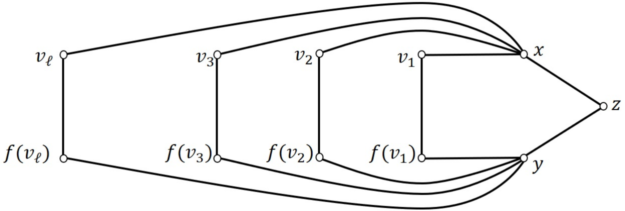

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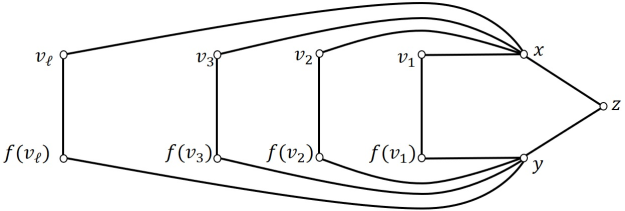

Theorem 3.1 (continued 1)

Proof (continued). Now $f(v_i) \neq f(v_j)$ for $i \neq j$ because of the hypothesis that any two vertices (x and v_i here) have exactly one common neighbor. So f is a one-to-one (injective) mapping from $N(x) \setminus \{z\}$ to $N(y) \setminus \{z\}$. Since $|N(x) \setminus \{z\}| = d(x) \geq d(y) = |N(y) \setminus \{z\}|$ (since x and y are nonadjacent and $d(x) \geq d(y)$ here) then f is actually onto (surjective) and we have $d(x) = d(y)$. Since x and y are arbitrary nonadjacent vertices of G, then any two nonadjacent vertices of G have the same degree. Equivalently, any two adjacent vertices in the complement of G, $\overline{G}$, have the same degree (since $d_{\overline{G}}(v) = n - 1 - d_G(v)$ for all $v \in V$). From this it follows that if $\overline{G}$ is connected then $\overline{G}$ (and hence G) is regular. Since we assumed $\Delta < n - 1$ then $\delta(\overline{G}) = n - 1 - \Delta(G) > 0$ and so $\overline{G}$ has no vectors of degree 0; that is, $\overline{G}$ has no singleton components. Now $\overline{G}$ cannot have two components each on two or more vertices, for then G would have a 4-cycle in violation of the hypothesis of any two vertices having exactly one common neighbor.

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Theorem 3.1 (continued 1)

Proof (continued). Since $\overline{G}$ cannot have components of size 1 nor two or more components of size 2 or greater, then $\overline{G}$ must have just one component; that is $\overline{G}$, must be connected. Therefore, as argued above, $\overline{G}$ (and hence G) is regular, say G is k-regular. As shown in the proof of Theorem 2.2 (see equation $(*)$), the number of 2-paths in graph $G = (V, G)$ is $\sum_{v \in V} \binom{d(v)}{2}$, so for G k-regular there are $|V|\binom{k}{2} = n\binom{k}{2}$ 2-paths. Under our hypothesis, every pair of vertices of G determines a unique 2-path (the path through the one common neighbor). So the number of 2-paths also equals $\binom{n}{2}$ and we have $n\binom{k}{2} = \binom{n}{2}$ or $\frac{nk(k-1)}{2} = \frac{n(n-1)}{2}$ or $k^2 - k = n - 1$ or $n = k^2 - k + 1$.

Theorem 3.1 (continued 2)

Proof (continued). Let $\mathbf{A}$ be the adjacency matrix of G. Then by Exercise 3.1.A, $\mathbf{A}^2 = \mathbf{J} + (k-1)\mathbf{I}$, where $\mathbf{J}$ is the $n \times n$ matrix all of whose vertices are 1 and $\mathbf{I}$ is the $n \times n$ identity matrix. The eigenvalues of $\mathbf{J}$ are 0 (with algebraic multiplicity $n-1$) and n (with algebraic multiplicity 1) by Lemma 3.1.A. Now

$$\det(\mathbf{A}^2 - \lambda\mathbf{I}) = \det(\mathbf{J} + (k-1)\mathbf{I} - \lambda\mathbf{I}) = \det(\mathbf{J} - (\lambda - (k-1))\mathbf{I}).$$

So we can convert eigenvalues of $\mathbf{J}$ to eigenvalues of $\mathbf{A}^2$ by adding $k-1$. So the eigenvalues of $\mathbf{A}^2$ are $k-1$ (with algebraic multiplicity $n-1$) and $n+k-1$ (with algebraic multiplicity 1). Notice $n = k^2 - k + 1$ from the previous paragraph, so $n + k - 1 = k^2$ is the eigenvalue of $\mathbf{A}^2$ of algebraic multiplicity 1. Recall that if λ is an eigenvalue of square matrix $\mathbf{A}$ then λ^k is an eigenvalue of $\mathbf{A}^k$ (see Theorem 5.1(1) in my online notes for Linear Algebra [MATH 2010] on 5.1. Eigenvalues and Eigenvectors).

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Theorem 3.1 (continued 3)

Proof (continued). Therefore the eigenvalues of $\mathbf{A}$ (and so the eigenvalues of G) are $\pm\sqrt{k-1}$ (with total algebraic multiplicity of $n-1$), and k with multiplicity 1 (we have k as an eigenvalue, not $-k$, by Exercise 1.1.22a(ii)). Because G is simple (so that there are no loops) then the diagonal entries of $\mathbf{A}$ are all 0 and so $\text{trace}(\mathbf{A}) = 0$ (the trace of a matrix is the sum of the diagonal entries; see my online notes for Theory of Matrices [MATH 5090] on 3.1. Basic Definitions and Notation). Now $\text{trace}(\mathbf{A})$ equals the sum of the eigenvalues of $\mathbf{A}$ (see Theorem 3.8.6 in the notes on 3.8. Eigenanalysis; Canonical Factorizations), so we must have $k - t\sqrt{k-1} = 0$ for some $t \in \mathbb{Z}$ (since the eigenvalues of A are k, $+\sqrt{k-1}$, and $-\sqrt{k-1}$ as described above); that is, $t\sqrt{k-1} = k$.

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Proof (continued). So $t^2(k-1) = k^2$ or

$$t^2 = \frac{k^2}{(k-1)^2} = \left(\frac{k}{k-1}\right)^2 = \left(\frac{k-1+1}{k-1}\right)^2 = \left(1 + \frac{1}{k-1}\right)^2 \in \mathbb{N}$$

so we must have $k = 2$ (and $t = 2$). Since $n = k^2 - k + 1$ then $k = 2$ implies $n = 3$. But this CONTRADICTS our assumption that $\Delta < n - 1$ since $\Delta = 2$ (because G is k-regular and we have $k = 2$). So the assumption is false and $\Delta = n - 1$ and there is a vertex of G of degree $n - 1$. $\square$