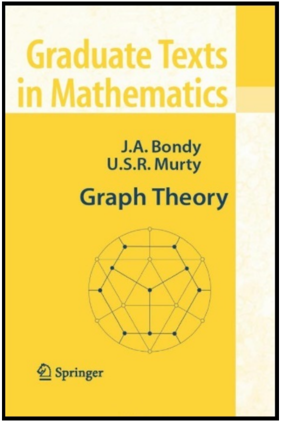
# Graph Theory

**Chapter 3. Connected Graphs**

3.2. Cut Edges—Proofs of Theorems



# Table of contents

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**Proposition 3.2.** An edge $e$ of a graph $G$ is a cut edge of a graph $G$ if and only if $\{e\}$ belongs to no cycle of $G$.

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Suppose $e = xy$ is not a cut edge of $G$. Then $x$ and $y$ belong to the same component of $G \setminus e$ (and $c(G) = c(G \setminus e)$). So by Exercise 3.1.4 (with $X = \{x\}$ and $Y = \{y\}$ in the notation of the exercise) there is an $xy$-path $P$ in $G \setminus e$. Then $P + e$ is a cycle in $G$ containing edge $e$. That is, if $e$ is not a cut edge of $G$ then $e$ is in some cycle of $G$. □

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