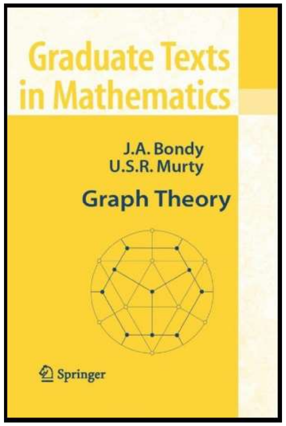
# Graph Theory

**Chapter 3. Connected Graphs**

3.3. Euler Tours—Proofs of Theorems








## Lemma 3.3.A

**Lemma 3.3.A.** If $G$ is an Eulerian graph then $G$ is even.

**Proof.** Let the walk $W = v_0 e_1 v_1 e_2 v_2 \cdots v_{m-1} e_m v_0$ be an Euler tour of $G$. The internal vertices of $W$, namely $v_1, v_2, \ldots, v_{m-1}$, are incident with edges in $W$ 2 at a time; for $i = 1, 2, \ldots, m-1$ we have $v_i$ is incident with edges $e_i$ and $e_{i+1}$. Since the edges are distinct in an Euler tour then, each internal vertex in an Euler tour (i.e., each internal vertex of $W$) is of even degree. Now $v_0$ may also appear in the set of vertices $\{v_1, v_2, \ldots, v_{m-1}\}$ and it will be incident to an even number of edges in the counting process used for these vertices. But $v_0$ is also incident to edges $e_1$ and $e_m$, so its total degree is even as well. That is (since $G$ is connected, by the definition of "tour"), each vertex of $G$ is of even degree and $G$ is an even graph, as claimed. □





## Theorem 3.4

**Theorem 3.4.** If $G$ is a connected even graph, then the walk $W$ returned by Fleury's Algorithm is an Euler tour of $G$.

**Proof.** Since the algorithm chooses an edge to add to the walk $W$ under construction and then deletes that edge (when replacing $F$ by $F \setminus e$) from those which may be chosen in subsequent steps, then the edges of walk $W$ must be distinct and so the walk is a trail throughout the procedure. The algorithm starts with initial vertex $u$ of $W$, so in graph $F$ we have $d_F(u) = d_G(u) - 1$ initially and remains so unless the walk returns to $u$ and continues on, so that $d_F(u)$ drops by 2 and $d_F(u)$ remains odd. Since $G$ is an even graph by hypothesis, the algorithm cannot terminate at some $x \neq u$ since such vertex $x$ is of even degree in $G$ and when $W$ has $x$ as its terminal vertex we then have $d_F(x)$ odd so that $\partial_F(x) \neq \varnothing$ and the algorithm does not end. So the algorithm and the walk $W$ produced can only terminate at vertex $u$. Hence the algorithm produces a closed trail of $G$ with vertex $u$ as its initial and terminal vertex. We now need to confirm that $W$ includes all edges of $G$. We do so with a proof by contradiction.



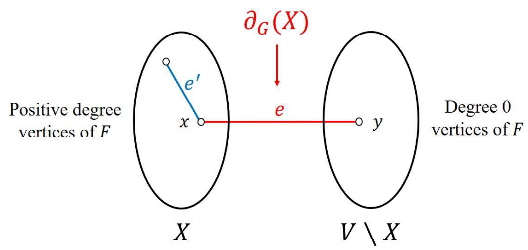


## Theorem 3.4 (continued 1)

**Proof (continued).** ASSUME that $W$, the walk produced by Fleury's Algorithm, is not an Euler tour of $G$. Let $X$ be the set of vertices of positive degree in subgraph $F$ at the stage when the algorithm terminates. Then $X \neq \varnothing$ (since $W$ is assumed to omit some edge(s) of $G$), $G$ is even by hypothesis, and $W$ determines an even induced subgraph of $G$, so the induced subgraph $F[X]$ is an even subgraph of $G$. As described above, the algorithm must terminate at vertex $u$ so $d_F(u) = 0$, $u \notin X$, and so $u \in V \setminus X$ so that $V \setminus X \neq \varnothing$. Now $\partial_G(X) \neq \varnothing$, or else $X$ and $V \setminus X$ would form a "separation" of $G$, but $G$ is hypothesized to be connected. But $\partial_F(X) = \varnothing$ since each vertex of $V \setminus X$ has degree 0 in $F$. Since the algorithm selects edges for inclusion in $W$ (and then deletes those edges in the creation of $F$), all of the edges of $\partial_G(X)$ must have been chosen for $W$ since, when the algorithm ended, we had $\partial_F(X) = \varnothing$. Let $e = xy$ be the last edge of $\partial_G(X)$ chosen for inclusion in $W$, where $x \in X$ and $y \in V \setminus X$.

## Theorem 3.4 (continued 2)

**Proof (continued).** At the step when $e$ was chosen, graph $F$ must have included edge $e$ and edge $e$ must be a cut edge of $F$ (since this is the step at which the last edge of $\partial_G(X)$ was added to $W$ so that $c(F \setminus e) = c(F) + 1$ because $F \setminus e$ has a "new" component which is a subset of $V \setminus X$ and includes $y$; see the figure below).




But when the algorithm ended, the degree of $x$ in the final graph $F$ was positive so that there was another choice of an edge $e'$ to add to $W$ when cut edge $e$ was chosen, since $d_F(x) > 0$ when the algorithm ends.





## Theorem 3.4 (continued 3)

**Theorem 3.4.** If $G$ is a connected even graph, then the walk $W$ returned by Fleury's Algorithm is an Euler tour of $G$.

**Proof (continued).** As argued above, all vertices of $X$ are of even positive degree in $F[X]$ when the algorithm ends. So by Exercise 3.2.3(a), the connected component of $F \setminus e$ containing vertex $x$ has no cut edges so that $e'$ is not a cut edge of $F \setminus e$ (and so not of $F$ itself before edge $e$ was chosen). But this violates the algorithm since it will not add cut edge $e$ to $W$ since non-cut edge $e'$ is available for inclusion in $W$, a CONTRADICTION. So the assumption that the walk produced by the algorithm is not an Euler tour is false. Hence, an Euler tour of $G$ is produced by the algorithm, as claimed. □