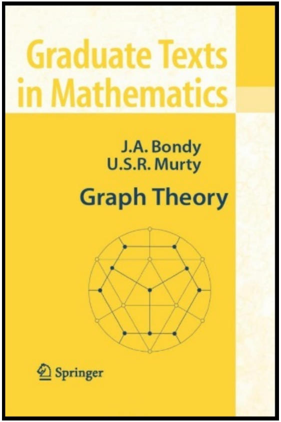
Graph Theory

Chapter 3. Connected Graphs

3.3. Euler Tours—Proofs of Theorems



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Lemma 3.3.A

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Theorem 3.4 (continued 1)

Proof (continued). ASSUME that W, the walk produced by Fleury's Algorithm, is not an Euler tour of G. Let X be the set of vertices of positive degree in subgraph F at the stage when the algorithm terminates. Then $X \neq \varnothing$ (since W is assumed to omit some edge(s) of G), G is even by hypothesis, and W determines an even induced subgraph of G, so the induced subgraph $F[X]$ is an even subgraph of G. As described above, the algorithm must terminate at vertex u so $d_F(u) = 0$, $u \notin X$, and so $u \in V \setminus X$ so that $V \setminus X \neq \varnothing$. Now $\partial_G(X) \neq \varnothing$, or else X and $V \setminus X$ would form a "separation" of G, but G is hypothesized to be connected. But $\partial_F(X) = \varnothing$ since each vertex of $V \setminus X$ has degree 0 in F. Since the algorithm selects edges for inclusion in W (and then deletes those edges in the creation of F), all of the edges of $\partial_G(X)$ must have been chosen for W since, when the algorithm ended, we had $\partial_F(X) = \varnothing$. Let $e = xy$ be the last edge of $\partial_G(X)$ chosen for inclusion in W, where $x \in X$ and $y \in V \setminus X$.

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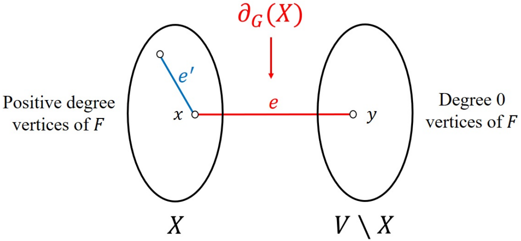
Theorem 3.4 (continued 2)

Proof (continued). At the step when e was chosen, graph F must have included edge e and edge e must be a cut edge of F (since this is the step at which the last edge of $\partial_G(X)$ was added to W so that $c(F \setminus e) = c(F) + 1$ because $F \setminus e$ has a "new" component which is a subset of $V \setminus X$ and includes y; see the figure below).



But when the algorithm ended, the degree of x in the final graph F was positive so that there was another choice of an edge e' to add to W when cut edge e was chosen, since $d_F(x) > 0$ when the algorithm ends.

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Proof (continued). As argued above, all vertices of X are of even positive degree in $F[X]$ when the algorithm ends. So by Exercise 3.2.3(a), the connected component of $F \setminus e$ containing vertex x has no cut edges so that e' is not a cut edge of $F \setminus e$ (and so not of F itself before edge e was chosen). But this violates the algorithm since it will not add cut edge e to W since non-cut edge e' is available for inclusion in W, a CONTRADICTION. So the assumption that the walk produced by the algorithm is not an Euler tour is false. Hence, an Euler tour of G is produced by the algorithm, as claimed. □

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