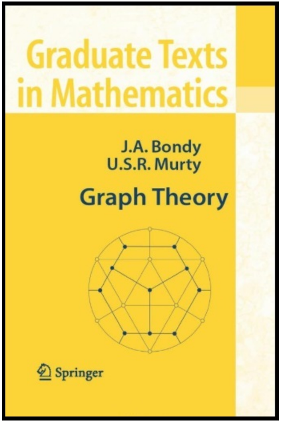
Graph Theory

Chapter 3. Connected Graphs

3.5. Cycle Double Covers—Proofs of Theorems



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Proof. Let family $\mathcal{C} = \{C_1, C_2, \ldots, C_\ell\}$ be a cycle covering of a graph G in which each edge of G is covered at most twice. The symmetric difference

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is the set of edges of G which are covered just once by $\mathcal{C}$ (associativity of the symmetric difference is shown in Exercise 2.6.2(a)).

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