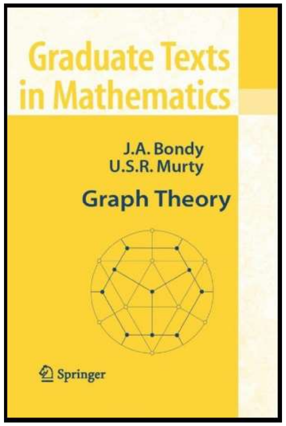
Graph Theory

Chapter 4. Trees

4.1. Forests and Trees—Proofs of Theorems



Proposition 4.1

Proposition 4.1. In a tree, any two vertices are connected by exactly one path.

Proof. Since a tree is connected, then by Exercise 3.1.4 any two vertices are connected by at least one path. ASSUME there are two or more distinct paths connecting two vertices. Then by Exercise 2.2.12 the tree contains a cycle, a CONTRADICTION to the definition of tree. So the assumption is false and there is exactly one path connecting any two vertices, as claimed. □

Theorem 4.3

Theorem 4.3. If T is a tree, then $e(T) = v(T) - 1$.

Proof. We give a proof by induction based on $v(T)$. When $v(T) = 1$, T is the trivial tree with no edges so that $e(T) = 0$ (notice that a tree cannot have any loops since a loop is a cycle of length 1), the claim holds for $v(T) = 1$, and this gives the base case.

Now suppose the result holds for all trees on k vertices and consider T a tree on $v(T) = k + 1 \geq 2$ vertices. By Proposition 4.2, T has a leaf w. Since w is a leaf then $d(w) = 1$ and so $uw \in E(T)$ for some $u \in V(T)$. Consider $T - w$. Since T is acyclic and $T - w$ is a subgraph of T then $T - w$ is acyclic. ASSUME $T - w$ is not connected. Then there are nonempty subsets X and Y of $V(T - w)$ such that no edge of $T - w$ has one end in X and one end in Y (i.e., X and Y are a "separation" of $T - w$). If $u \in X$ then $X \cup \{w\}$ and Y form a separation of T so that T is not connected, a CONTRADICTION. If $u \in Y$ then X and $Y \cup \{w\}$ form a separation of T so that T is not connected, a CONTRADICTION.

Theorem 4.3 (continued)

Theorem 4.3. If T is a tree, then $e(T) = v(T) - 1$.

Proof (continued). So the assumption that $T - w$ is not connected is false. So $T - w$ is a connected acyclic graph; i.e., $T - w$ is a tree. By construction,

$$v(T - w) = v(T) - 1 = k \text{ and } e(T - w) = e(T) - 1. \qquad (*)$$

By the induction hypothesis $k - 1 = e(T - w) = v(T - w) - 1$. Therefore we have by $(*)$ that $k - 1 = e(T) - 1 = (v(T) - 1) - 1$ or $k = e(T) = v(T) - 1$ where $v(T) = k + 1$. So the result holds for $v(T) = k + 1$ and hence by mathematical induction $e(T) = v(T) - 1$ for any tree T, as claimed. □

Corollary 4.1.B

Corollary 4.1.B. Rédei's Theorem.
Every tournament has a directed Hamilton path.

Proof. Let $v_1, v_2, \ldots, v_n$ be a median order of the tournament. By Theorem 4.1.A(M1), with $1 \le i \le n-1$ and $j = i+1$ we have that v_i, v_{i+1} is a median order on $T[\{v_i, v_{i+1}\}]$. But $T[\{v_i, v_{i+1}\}]$ is just an arc between v_i and v_{i+1}, so with v_i, v_{i+1} as a median order then $T[\{v_i, v_{i+1}\}]$ must consist of arc (v_i, v_{i+1}). So $P = (v_1, v_2, \ldots, v_n)$ is a directed Hamilton path from vertex v_1 to vertex v_n. □

Theorem 4.5

Theorem 4.5. Any tournament on $2k$ vertices contains a copy of each branching on $k+1$ vertices.

Proof. Let $v_1, v_2, \ldots, v_k$ be a median order of a tournament T on $2k$ vertices. Let B be any branching on $k+1$ vertices (notice that B is not given as being in T). Consider the intervals $v_1, v_2, \ldots, v_i$ where $1 \le i \le 2k$. We show something slightly more general than the conclusion of the theorem. We show by induction on k that there is a copy of B in T (establishing the theorem) with the additional property that the vertex set of the copy of B includes at least half the vertices in any such interval.

With $k = 1$, B is a single are between $k+1 = 2$ vertices. Consider the subinterval v_1, v_2 of the median interval. By Theorem 4.1.A(M1), arc (v_1, v_2) is in T and so we take the subgraph of T induced by arc (v_1, v_2) as a copy of B and the result holds for $k = 1$. This establishes the base case.

Theorem 4.5 (continued 1)

Proof (continued). Now let $k \ge 2$ and suppose that the claim holds for all tournaments on $2(k-1)$ vertices and for all B a branching on k vertices. Let T be a tournament on $2k$ vertices and let $v_1, v_2, \ldots v_{2k}$ be a median order of T. Since B is a branching then it has some vertex y of indegree 1 and outdegree 0 (i.e., B contains the leaf y). Define $B' = B - y$ so that B' is a branching on k vertices. Define $T' = T[\{v_1, v_2, \ldots, v_{2k-2}\}] = T - \{v_{2k-1}, v_{2k}\}$ so that T' is a tournament on $2k-2$ vertices. By Theorem 2.7.A(M1) (with $i = 1$ and $j = 2k-2$), $v_1, v_2, \ldots, v_{2k-2}$ is a median order of T'. So by the induction hypothesis there is a copy of B' in T' whose vertex set includes at least half of the vertices of any interval of the form $v_1, v_2, \ldots, v_i$ where $1 \le i \le 2k-2$. Let x be the (unique) predecessor of y (the leaf) in B. Suppose for the sake of notation that x is located at vertex v_{i^*} of T'.

Theorem 4.5 (continued 2)

Proof (continued). In T, by Theorem 4.1.A(M2), v_{i^*} dominates at least half of the vertices $v_{i^*+1}, v_{i^*+2}, \ldots, v_{2k}$ (a list of $2k - i^*$ vertices) so that

$$v_{i^*} \text{ dominates at least } k - 1/2 \text{ of the vertices } v_{i^*+1}, v_{i^*+2}, \ldots, v_{2k}. \quad (*)$$

On the other hand, B' includes at least $(i^* - 1)/2$ of the $i^* - 1$ vertices $v_1, v_2, \ldots, v_{i^*-1}$ by the induction hypothesis.

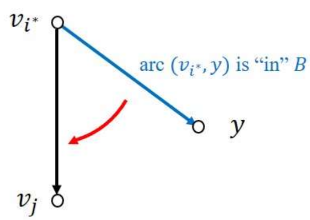
$$B' \text{ includes at most } (k-1) - (i^* - 1)/2 = k - (i^* + 1)/2$$

$$\text{of the vertices } v_{i^*+1}, v_{i^*+2}, \ldots, v_{2k} \quad (**)$$

(we take $k-1) - (i^* - 1)/2$ since B' includes at least $(i^* \ge 1)$, so $(*)$ and $(**)$ imply that v_{i^*} dominates some vertex v_j where $i^* + 1 \le j \le 2k$ where v_j is not in B'. So add vertex v_j to B' and add arc (v_{i^*}, v_j) and vertex v_j to B' to get a copy of B in tournament T.

Theorem 4.5 (continued 3)

Proof (continued). We have really just deleted the arc (v_{i^*}, y) in B to produce B', a copy of which is in T' by the induction hypotheses, and the replaces (v_{i^*}, y) with arc (v_{i^*}, v_j) in T:



Now let $1 \leq i \leq 2k$ and consider the interval $v_1, v_2, \ldots, v_i$. For $1 \leq i \leq j-1$ the copy of B' contains at least half of the vertices $v_1, v_2, \ldots, v_i$ and so does B (since the subgraphs induced by the corresponding vertices of B and B' are the same on these vertices).

Theorem 4.5 (continued 4)

Theorem 4.5. Any tournament on $2k$ vertices contains a copy of each branching on $k+1$ vertices.

Proof (continued).
For $j \leq i \leq 2k-2$ the copy of B' contains at least half of the vertices in $\{v_1, v_2, \ldots, v_i\} \setminus \{v_j\}$ (so at least $(i-1)/2$ of these vertices) and so B contains these plus vertex v_j and so B contains at least $(i-1)/2 + 1 = (1+1)/3 \geq i/2$ of $\{v_1, v_2, \ldots, v_i\}$. When $i = 2k-1$ or $i = 2k$, again B' contains at least $(2k-2)/2 = k-1$ of $\{v_1, v_2, \ldots, v_i\} \setminus \{v_j\}$ and so B contains at least $(k-1)+1 = k$ of the vertices $\{v_1, v_2, \ldots, v_i\}$, as claimed. So by claim holds for any tournament on $2k$ vertices and hence, by mathematical induction, holds for all tournaments on $2k$ vertices where $k \in \mathbb{N}$. □