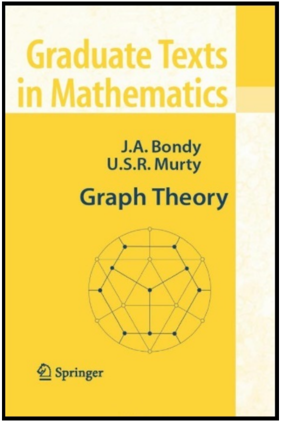
Graph Theory

Chapter 4. Trees

4.2. Spanning Trees—Proofs of Theorems



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Proposition 4.6

Proposition 4.6. A graph G is connected if and only if G has a spanning tree.

Proof. Suppose graph G has a spanning tree T. Then G is connected since for any two vertices in G (and hence in T) there is a unique path in T joining the vertices, by Proposition 4.1. So for any two vertices in G (on any two nonempty sets of vertices X and Y) there is a path in G joining the vertices (and there is an (X, Y)-path in G). So by Exercise 3.1.4, G is connected.

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Suppose graph G is connected. If G is a tree, then it trivially has a spanning tree. If G is not a tree, then it is not acyclic and so G contains some cycle. If e is an edge of this cycle then by Proposition 3.2, e is not a cut edge of G and by the definition of "cut edge," $1 = c(G) = c(G \setminus e)$. That is, $G \setminus e$ is connected. We can iterate this process of removing edges $e_1, e_2, \ldots, e_\ell$ from cycles in G until $(\cdots((G \setminus e_1) \setminus e_2) \setminus \cdots \setminus e_\ell)$ is acyclic and connected. So $T = (\cdots((G \setminus e_1) \setminus e_2) \setminus \cdots \setminus e_\ell)$ is a spanning subtree of G. □

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Theorem 4.7

Theorem 4.7. A graph is bipartite if and only if it contains no odd cycle.

Proof. A (not necessarily connected) graph is bipartite if and only if each of its components is bipartite. A (not necessarily connected) graph contains an odd cycle if and only if one of its components contains an odd cycle. So, without loss of generality, we can address the claim for connected graphs.

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Let $G[X, Y]$ be a connected bipartite graph. Then the vertices of any path in G are elements alternatively (remember that a path forms a "linear sequence" of vertices) of sets X and Y. Thus all paths connecting vertices in different parts are of odd length and all paths connecting vertices in the same part are of even length. If C is any cycle in G and e is an edge of C, then one end of e is in X and one end of e is in Y, since G is bipartite. So $C \setminus e$ is a path between an element of X and an element of Y; that is $C \setminus e$ is an odd length path ($C \setminus e$ contains an odd number of edges) and so C is an even cycle. So any cycle in G must be an even cycle. That is, G contains no odd cycle.

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Theorem 4.7 (continued)

Proof (continued). Suppose G is a connected graph that contains no odd cycles. By Theorem 4.6, G has a spanning tree T. Let $v \in V(T)$. As in Note 4.2.A, let $X = \{x \in V(T) \mid d_T(x, v) \text{ is even}\}$ and $Y = \{y \in V(T) \mid d_T(y, v) \text{ is odd}\}$. Then (X, Y) is a bipartition of T. We claim that it is also a bipartition of G.

Let $e = uv$ be an arbitrary edge of G. If e is an edge of T then, since (X, Y) is a bipartition of T, one end of e is in X and the other end of e is in Y. If e is not an edge of T then, since T is a spanning tree and by Proposition 4.1, there is a unique uv-path in T, denoted $P = uTv$ (in the notation of Diestel introduced in Section 4.1). So $P + e$ is a cycle and so is even by hypothesis. Hence path P is odd and by the argument above one end of path P is in X and the other end of P is in Y. That is, one end of edge e is in X and the other end of edge e is in Y. Since e is an arbitrary edge of G, then (X, Y) is a bipartition of graph G and G is bipartite, as claimed. □

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Theorem 4.8

Theorem 4.8. Cayley's Formula.
The number of labeled trees on n vertices is n^{n-2}.

Proof. We count the number of labeled branchings on n vertices and show that this is n^{n-1}. Now a labeled branching is determined by its (unique) underlying graph and the choice of a root (there are n choices for the root). So each labeled tree on n vertices corresponds to n branchings. Hence it will follow that the number of labeled trees on n vertices is $n^{n-1}/n = n^{n-2}$.

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We now present a procedure to produce a labeled branching on n vertices. We start with n vertices and no arcs. We then introduce one arc at a time in such a way that at each step we have a branching forest. In doing so, each time we add a new arc, the number of components decreases by 1 (since we either (1) connect an isolated vertex to the root of a branching, (2) connect a vertex in a branching to an isolated vertex, or (3) connect a vertex in a branching to the root of another branching; ...

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Theorem 4.8 (continued 1)

Proof (continued). ...in each case, we have joined two components of the underlying graph). If there are k components at some stage then there are $n(k-1)$ ways to choose an arc (u, v) to add: the tail u of (u, v) can be any of the n vertices and then the head can be any root of a component of the branching forest which does not contain u (and there are $k-1$ such components). Notice that choosing a pair of vertices u and v determines the direction of the arc (uv) (this is why Bondy and Murty refer to "choosing edges"). We start with $k = n$ (when the components are n isolated vertices) and end after the step corresponding to $k = 2$ (after this step there is one component, as desired). So the number of ways of constructing a branching on n vertices with this procedure is:

$$\prod_{i=1}^{n-1} n(n-i) = n^{n-1}(n-1)! \qquad (*)$$

(here the number of components is $n - i = k - 1$ so that $i = n - k + 1$ and $k = n$ implies $i = 1$, and $k = 2$ implies $i = n - 1$).

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Theorem 4.8 (continued 2)

Theorem 4.8. Cayley's Formula.
The number of labeled trees on n vertices is n^{n-2}.

Proof (continued). Now the procedure produces every labeled branching on n vertices, but the order in which the arcs are chosen is irrelevant and this procedure produces a given branching in $(n-1)!$ ways (the number of ways the $n-1$ arcs can be ordered). So from $(*)$ we have that the total number of labeled branchings on n vertices is
$n^{n-1}(n-1)!/(n-1)! = n^{n-1}$. Therefore, as explained above, this implies that the number of labeled trees is $n^{n-1}/n = n^{n-2}$, as claimed. $\square$