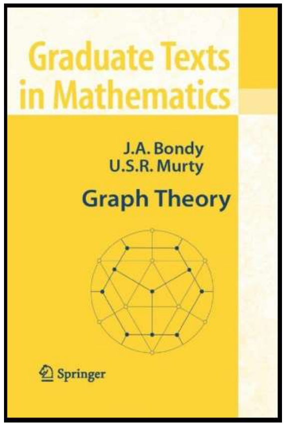
Graph Theory

Chapter 4. Trees

4.3. Fundamental Cycles and Bonds—Proofs of Theorems



Theorem 4.10

Theorem 4.10. Let T be a spanning tree of a connected graph G, and let S be a subset of its cotree $\overline{T}$. Then the symmetric difference of the fundamental cycles $C = \triangle\{C_e \mid e \in S\}$ is an even subgraph of G. Moreover, $C \cap \overline{T} = S$, and C is the only even subgraph of G with this property.

Proof. Each fundamental cycle C_e is an even subgraph of G, so the symmetric difference $C = \triangle\{C_e \mid e \in S\}$ is also an even subgraph by Corollary 2.16. Next, each edge of $S \subseteq \overline{T}$ occurs in exactly one fundamental cycle (namely, $e \in S$ is in C_e only, since a fundamental cycle consists of a single edge in $\overline{T}$ and a path in T). So each edge of S is in $\triangle\{C_e \mid e \in S\}$ and hence $C \cap \overline{T} = S$, as claimed.

To show that C is the only such even subgraph of G, let C' be any even subgraph of G such that $C' \cap \overline{T} = S$.

Theorem 4.10 (continued 1)

Proof (continued). Then

$$\begin{aligned}
(C \triangle C') \cap \overline{T} &= ((C \setminus C') \,\dot\cup\, (C' \setminus C)) \cap \overline{T} \\
&= ((C \setminus C') \cap \overline{T}) \,\dot\cup\, ((C' \setminus C) \cap \overline{T}) \\
&= (C \cap \overline{C'} \cap \overline{T}) \,\dot\cup\, (C' \cap \overline{C} \cap \overline{T}) \text{ using overlines} \\
&\quad \text{ to represent complements} \\
&= ((C \cap \overline{T} \cap \overline{C'}) \cup (C \cap \overline{T} \cap T)) \\
&\quad \cup ((C' \cap \overline{T} \cap \overline{C}) \cup (C' \cap \overline{T} \cap T)) \text{ since } \overline{T} \cap T = \varnothing \\
&= ((C \cap \overline{T}) \cap (\overline{C'} \cup T)) \cup ((C' \cap \overline{T}) \cap (\overline{C} \cup T)) \\
&= ((C \cap \overline{T}) \cap \overline{(C' \cap \overline{T})} \cup ((C' \cap \overline{T}) \cap \overline{(C \cap \overline{T})}) \\
&\quad \text{ by de Morgan's Laws} \\
&= ((C \cap \overline{T}) \setminus (C' \cap \overline{T})) \cup ((C' \cap \overline{T}) \setminus (C \cap \overline{T})) \\
&= (C \cap \overline{T}) \triangle (C' \cap \overline{T}) = S \triangle S = \varnothing.
\end{aligned}$$

Theorem 4.10 (continued 2)

Theorem 4.10. Let T be a spanning tree of a connected graph G, and let S be a subset of its cotree $\overline{T}$. Then the symmetric difference of the fundamental cycles $C = \triangle\{C_e \mid e \in S\}$ is an even subgraph of G. Moreover, $C \cap \overline{T} = S$, and C is the only even subgraph of G with this property.

Proof (continued). That is, $(C \triangle C') \cap \overline{T} = \varnothing$ and so $C \triangle C' \subseteq T$. But T is a tree and so the only even subgraph of a tree is the even empty subgraph (by, say, Veblen's Theorem, Theorem 2.7), so $C \triangle C' = \varnothing$. That is, $C = C'$ and C is the only even subgraph of G such that $C \cap \overline{T} = S$, as claimed. □

Corollary 4.11

Corollary 4.11. Let T be a spanning tree of a connected graph G. Every even subgraph of G can be expressed uniquely as a symmetric difference of fundamental cycles with respect to T.

Proof. Let C be an even subgraph of G. Define $S = C \cap \overline{T}$. By Theorem 4.10, we can take $C = \triangle\{C_e \mid e \in S\}$ and this is the only way of expressing even subgraph C as a symmetric difference of fundamental cycles with respect to T, as claimed. □

Corollary 4.12

Corollary 4.12. Every cotree of a connected graph (that is, every complement of a spanning tree) is contained in a unique even subgraph of the connected graph.

Proof. Let T be a spanning tree of the connected graph and let $S = \overline{T}$. Then by Theorem 4.10, there is a unique even subgraph C such that $C \cap \overline{T} = S = \overline{T}$ (in fact, the unique even subgraph is $C = \triangle\{C_e \mid e \in S\}$). That is, there is a unique even subgraph C such that $\overline{T} \subseteq C$, as claimed. □

Lemma 4.3.A

Lemma 4.3.A. Let G be a connected graph, T a spanning tree in G, and $e = xy$ an edge of T. Let X be the vertex set of the component of $T \setminus e$ which contains x. Then the bond $B_e = \partial(X)$ is the only bond of G contained in $\overline{T} \cup \{e\}$ that includes edge e.

Proof. Let B be any bond of G contained in $\overline{T} \cup \{e\}$ and including e. Since both B and B_e are also edge cuts, then by Corollary 2.12 we have $B \triangle B_e$ is an edge cut of G. But since both B and B_e contain edge e, then edge cut $B \triangle B_e$ does not contain e and hence $B \triangle B_e \subseteq \overline{T}$. But since T is a spanning tree of G then the only edge cut of G that is contained in $\overline{T}$ is the empty edge cut by Note 4.3.B. So $B \triangle B_e = \varnothing$ and hence $B = B_e$. That is, bond B_e as described is unique, as claimed. □