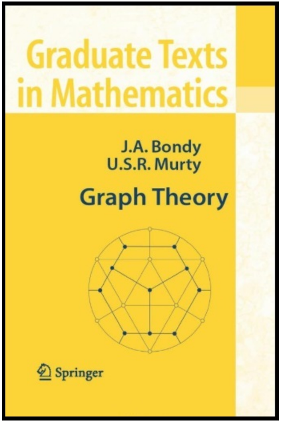
# Graph Theory

## Chapter 4. Trees

4.3. Fundamental Cycles and Bonds—Proofs of Theorems



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# Theorem 4.10

**Theorem 4.10.** Let $T$ be a spanning tree of a connected graph $G$, and let $S$ be a subset of its cotree $\overline{T}$. Then the symmetric difference of the fundamental cycles $C = \triangle\{C_e \mid e \in S\}$ is an even subgraph of $G$. Moreover, $C \cap \overline{T} = S$, and $C$ is the only even subgraph of $G$ with this property.

**Proof.** Each fundamental cycle $C_e$ is an even subgraph of $G$, so the symmetric difference $C = \triangle\{C_e \mid e \in S\}$ is also an even subgraph by Corollary 2.16. Next, each edge of $S \subseteq \overline{T}$ occurs in exactly one fundamental cycle (namely, $e \in S$ is in $C_e$ only, since a fundamental cycle consists of a single edge in $\overline{T}$ and a path in $T$). So each edge of $S$ is in $\triangle\{C_e \mid e \in S\}$ and hence $C \cap \overline{T} = S$, as claimed.

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# Theorem 4.10 (continued 1)

**Proof (continued).** Then

$$
\begin{aligned}
(C \triangle C') \cap \overline{T} &= ((C \setminus C') \,\dot\cup\, (C' \setminus C)) \cap \overline{T} \\
&= ((C \setminus C') \cap \overline{T}) \,\dot\cup\, ((C' \setminus C) \cap \overline{T}) \\
&= (C \cap \overline{C'} \cap \overline{T}) \,\dot\cup\, (C' \cap \overline{C} \cap \overline{T}) \text{ using overlines} \\
&\quad \text{ to represent complements} \\
&= ((C \cap \overline{T} \cap \overline{C'}) \cup (C \cap \overline{T} \cap T)) \\
&\quad \cup((C' \cap \overline{T} \cap \overline{C}) \cup (C' \cap \overline{T} \cap T)) \text{ since } \overline{T} \cap T = \varnothing \\
&= ((C \cap \overline{T}) \cap (\overline{C'} \cup T)) \cup ((C' \cap \overline{T}) \cap (\overline{C} \cup T)) \\
&= ((C \cap \overline{T}) \cap \overline{(C' \cap \overline{T})} \cup ((C' \cap \overline{T}) \cap \overline{(C \cap \overline{T})}) \\
&\quad \text{ by de Morgan's Laws} \\
&= ((C \cap \overline{T}) \setminus (C' \cap \overline{T})) \cup ((C' \cap \overline{T}) \setminus (C \cap \overline{T})) \\
&= (C \cap \overline{T}) \triangle (C' \cap \overline{T}) = S \triangle S = \varnothing.
\end{aligned}
$$

# Theorem 4.10 (continued 2)

**Theorem 4.10.** Let $T$ be a spanning tree of a connected graph $G$, and let $S$ be a subset of its cotree $\overline{T}$. Then the symmetric difference of the fundamental cycles $C = \triangle\{C_e \mid e \in S\}$ is an even subgraph of $G$. Moreover, $C \cap \overline{T} = S$, and $C$ is the only even subgraph of $G$ with this property.

**Proof (continued).** That is, $(C \triangle C') \cap \overline{T} = \varnothing$ and so $C \triangle C' \subseteq T$. But $T$ is a tree and so the only even subgraph of a tree is the even empty subgraph (by, say, Veblen's Theorem, Theorem 2.7), so $C \triangle C' = \varnothing$. That is, $C = C'$ and $C$ is the only even subgraph of $G$ such that $C \cap \overline{T} = S$, as claimed. □

# Corollary 4.11

**Corollary 4.11.** Let $T$ be a spanning tree of a connected graph $G$. Every even subgraph of $G$ can be expressed uniquely as a symmetric difference of fundamental cycles with respect to $T$.

**Proof.** Let $C$ be an even subgraph of $G$. Define $S = C \cap \overline{T}$. By Theorem 4.10, we can take $C = \triangle\{C_e \mid e \in S\}$ and this is the only way of expressing even subgraph $C$ as a symmetric difference of fundamental cycles with respect to $T$, as claimed. $\square$

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# Corollary 4.12

**Corollary 4.12.** Every cotree of a connected graph (that is, every complement of a spanning tree) is contained in a unique even subgraph of the connected graph.

**Proof.** Let $T$ be a spanning tree of the connected graph and let $S = \overline{T}$. Then by Theorem 4.10, there is a unique even subgraph $C$ such that $C \cap \overline{T} = S = \overline{T}$ (in fact, the unique even subgraph is $C = \triangle\{C_e \mid e \in S\}$). That is, there is a unique even subgraph $C$ such that $\overline{T} \subseteq C$, as claimed. □

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# Lemma 4.3.A

**Lemma 4.3.A.** Let $G$ be a connected graph, $T$ a spanning tree in $G$, and $e = xy$ an edge of $T$. Let $X$ be the vertex set of the component of $T \setminus e$ which contains $x$. Then the bond $B_e = \partial(X)$ is the only bond of $G$ contained in $\overline{T} \cup \{e\}$ that includes edge $e$.

**Proof.** Let $B$ be any bond of $G$ contained in $\overline{T} \cup \{e\}$ and including $e$. Since both $B$ and $B_e$ are also edge cuts, then by Corollary 2.12 we have $B \triangle B_e$ is an edge cut of $G$. But since both $B$ and $B_e$ contain edge $e$, then edge cut $B \triangle B_e$ does not contain $e$ and hence $B \triangle B_e \subseteq \overline{T}$. But since $T$ is a spanning tree of $G$ then the only edge cut of $G$ that is contained in $\overline{T}$ is the empty edge cut by Note 4.3.B. So $B \triangle B_e = \varnothing$ and hence $B = B_e$. That is, bond $B_e$ as described is unique, as claimed. □

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