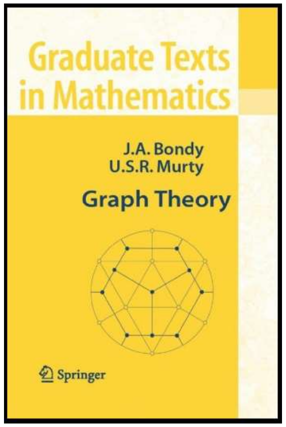
# Graph Theory

**Chapter 5. Nonseparable Graphs**

5.1. Cut Vertices—Proofs of Theorems








## Theorem 5.1

**Theorem 5.1.** A connected graph on three or more vertices has no cut vertices if and only if any two distinct vertices are connected by two internally disjoint paths.

**Proof.** Suppose $G$ is a connected graph such that any two distinct vertices are connected by two internally disjoint paths. Let $v$ be a vertex of $G$ and consider $G - v$. For any two distinct vertices $x$ and $y$ in $G - v$, there are two internally disjoint paths in $G$ connecting $x$ and $y$. Since vertex $v$ cannot be an internal vertex of both paths, then one of these paths must be in $G - v$. Since $x$ and $y$ are arbitrary vertices in $G - v$, then by Exercise 3.1.4 graph $G - v$ is connected. That is, $v$ is not a cut vertex of $G$. Since $v$ is an arbitrary vertex of graph $G$, then $G$ has no cut vertices, as claimed.

Now suppose $G$ is a connected graph on three or more vertices that has no cut vertices. Let $u$ and $v$ be two vertices of $G$. We prove by induction on the distance $d(u, v)$ that these vertices are connected by two internally disjoint paths.





## Theorem 5.1 (continued 1)

**Theorem 5.1.** A connected graph on three or more vertices has no cut vertices if and only if any two distinct vertices are connected by two internally disjoint paths.

**Proof (continued).** First, suppose $u$ and $v$ are adjacent so that $d(u, v) = 1$. Let $e$ be edge $uv$. Since neither $u$ nor $v$ is a cut vertex then, by Exercise 5.1.2, $e$ is not a cut edge. So by Proposition 3.2, edge $e$ lies in a cycle $C$ of $G$. So $u$ and $v$ are connected by the two internally disjoint paths $uev$ and $C \setminus e$, establishing the base case.

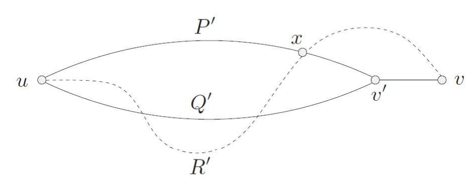
Second, suppose the claim holds for any two vertices at a distance less than $k$ where $k \geq 2$. Let $d(u, v) = k$. Consider a $uv$-path of length $k$ and let $v'$ be the immediate predecessor of $v$ on this path. Then $d(u, v') = k - 1$ (it cannot be less than this, or else $d(u, v)$ would be less than $k$). By the induction hypothesis, $u$ and $v'$ are connected by two internally disjoint paths, say $P'$ and $Q'$ (see Figure 5.2).





## Theorem 5.1 (continued 2)

**Proof (continued).**




Because $G$ has no cut vertices by hypothesis, then $G - v'$ is connected and therefore contains a $uv$-path, say $R'$ (by Exercise 3.1.4). Now path $R'$ meets $P' \cup Q'$ in possibly several points, but $R'$ definitely meets $P' \cup Q'$ at vertex $u$. Let $x$ be the last vertex of $R'$ at which $R'$ meets $P' \cup Q'$. Without loss of generality (for the sake of notation), say $x$ lies on $P'$. Define paths $P = uP'xR'v$ and $Q = uQ'v'v$. Then $P$ and $Q$ are internally disjoint $uv$-paths in $G$. So, by induction, for any two distinct vertices in $G$ (say the distance between these vertices is $n \in \mathbb{N}$) there are two internally disjoint paths joining the two vertices, as claimed. □