Graph Theory

Chapter 5. Nonseparable Graphs

5.1. Cut Vertices—Proofs of Theorems

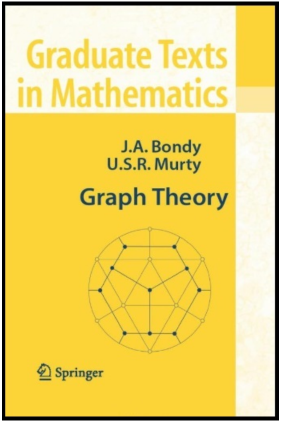

Table of contents

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Proof. Suppose G is a connected graph such that any two distinct vertices are connected by two internally disjoint paths. Let v be a vertex of G and consider $G - v$. For any two distinct vertices x and y in $G - v$, there are two internally disjoint paths in G connecting x and y. Since vertex v cannot be an internal vertex of both paths, then one of these paths must be in $G - v$. Since x and y are arbitrary vertices in $G - v$, then by Exercise 3.1.4 graph $G - v$ is connected. That is, v is not a cut vertex of G. Since v is an arbitrary vertex of graph G, then G has no cut vertices, as claimed.

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Proof (continued). First, suppose u and v are adjacent so that $d(u, v) = 1$. Let e be edge uv. Since neither u nor v is a cut vertex then, by Exercise 5.1.2, e is not a cut edge. So by Proposition 3.2, edge e lies in a cycle C of G. So u and v are connected by the two internally disjoint paths uev and $C \setminus e$, establishing the base case.

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Second, suppose the claim holds for any two vertices at a distance less than k where $k \geq 2$. Let $d(u, v) = k$. Consider a uv-path of length k and let v' be the immediate predecessor of v on this path. Then $d(u, v') = k - 1$ (it cannot be less than this, or else $d(u, v)$ would be less than k). By the induction hypothesis, u and v' are connected by two internally disjoint paths, say P' and Q' (see Figure 5.2).

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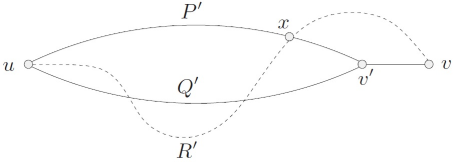

Because G has no cut vertices by hypothesis, then $G - v'$ is connected and therefore contains a uv-path, say R' (by Exercise 3.1.4). Now path R' meets $P' \cup Q'$ in possibly several points, but R' definitely meets $P' \cup Q'$ at vertex u. Let x be the last vertex of R' at which R' meets $P' \cup Q'$. Without loss of generality (for the sake of notation), say x lies on P'. Define paths $P = uP'xR'v$ and $Q = uQ'v'v$. Then P and Q are internally disjoint uv-paths in G. So, by induction, for any two distinct vertices in G (say the distance between these vertices is $n \in \mathbb{N}$) there are two internally disjoint paths joining the two vertices, as claimed. □

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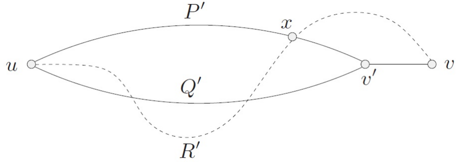

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