Graph Theory

Chapter 5. Nonseparable Graphs

5.2. Separations and Blocks—Proofs of Theorems

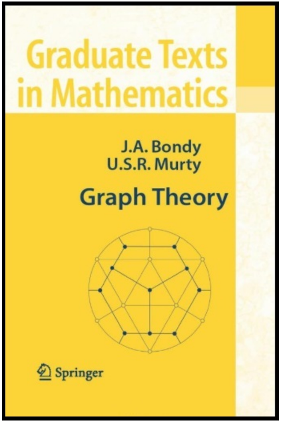


Table of contents

Theorem 5.2

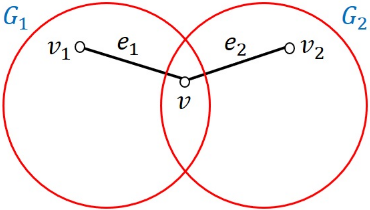
Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Proof. Suppose G is not nonseparable; that is, suppose G is separable. Then G can be decomposed into two nonempty connected subgraphs G_1 and G_2 which have just one vertex v in common. Let e_i be an edge of G_i incident with v, for $i \in \{1, 2\}$.

Theorem 5.2

Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

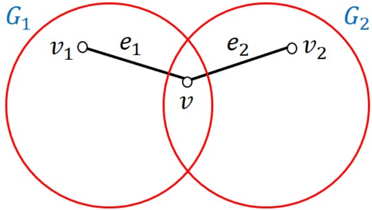
Proof. Suppose G is not nonseparable; that is, suppose G is separable. Then G can be decomposed into two nonempty connected subgraphs G_1 and G_2 which have just one vertex v in common. Let e_i be an edge of G_i incident with v, for $i \in \{1, 2\}$.



Theorem 5.2

Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Proof. Suppose G is not nonseparable; that is, suppose G is separable. Then G can be decomposed into two nonempty connected subgraphs G_1 and G_2 which have just one vertex v in common. Let e_i be an edge of G_i incident with v, for $i \in \{1, 2\}$.



Theorem 5.2 (continued 1)

Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Proof (continued). If either e_1 or e_2 is a loop then there is no cycle including both e_1 and e_2 (remember, a loop is a cycle of length 1). If neither e_1 nor e_2 is a loop then v is a cut vertex of G (here, the components of $G - v$ determine the decomposition of G required in the definition of "separation"). Let v_i be the other end of e_i for $i \in \{1, 2\}$. Then there is no $v_1 v_2$-path in $G - v$ (since v is a cut vertex). So there is no cycle in G containing the two edges e_1 and e_2 (or else the cycle minus vertex v would be a $v_1 v_2$-path). The contrapositive of what we have shown is: If any two edges lie on a common cycle then the connected graph is nonseparable.

Theorem 5.2 (continued 1)

Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Proof (continued). If either e_1 or e_2 is a loop then there is no cycle including both e_1 and e_2 (remember, a loop is a cycle of length 1). If neither e_1 nor e_2 is a loop then v is a cut vertex of G (here, the components of $G - v$ determine the decomposition of G required in the definition of "separation"). Let v_i be the other end of e_i for $i \in \{1, 2\}$. Then there is no v_1v_2-path in $G - v$ (since v is a cut vertex). So there is no cycle in G containing the two edges e_1 and e_2 (or else the cycle minus vertex v would be a v_1v_2-path). The contrapositive of what we have shown is: If any two edges lie on a common cycle then the connected graph is nonseparable.

Theorem 5.2 (continued 2)

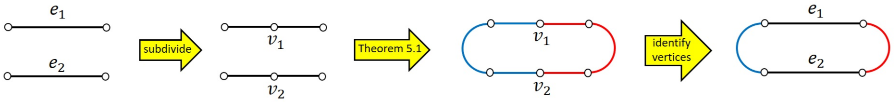
Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Proof (continued). Now suppose that G is nonseparable. Let e_1 and e_2 be two edges of G. Subdivide e_i by a new vertex v_i for $i \in \{1, 2\}$ producing graph H (we create H by subdividing edges of G to insure that H has at least three vertices; we'll apply Theorem 5.1 to H and at least three vertices are required). By Exercise 5.2.1, since G is nonseparable then so is H. Since every cut vertex of a graph is a separating vertex and H is nonseparable, then H has no cut vertices.

Theorem 5.2 (continued 3)

Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Proof (continued).

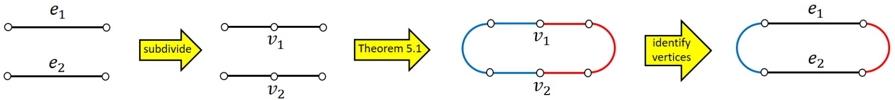


Then by Theorem 5.1 there are two internally disjoint v_1v_2-paths in H. These two paths form a cycle in H which contains v_1 and v_2. Now identify v_i with one end of e_i for $i \in \{1, 2\}$ in the cycle. This gives a cycle in G that contains edges e_1 and e_2. Since e_1 and e_2 are arbitrary edges of G, we have shown that if G is nonseparable then any two edges of G lie on a common cycle. □

Theorem 5.2 (continued 3)

Theorem 5.2. A connected graph is nonseparable if and only if any two of its edges lie on a common cycle.

Proof (continued).



Then by Theorem 5.1 there are two internally disjoint $v_1 v_2$-paths in H. These two paths form a cycle in H which contains v_1 and v_2. Now identify v_i with one end of e_i for $i \in \{1, 2\}$ in the cycle. This gives a cycle in G that contains edges e_1 and e_2. Since e_1 and e_2 are arbitrary edges of G, we have shown that if G is nonseparable then any two edges of G lie on a common cycle. $\square$

Proposition 5.3

Proposition 5.3. Let G be a graph. Then:

(a) any two blocks of G have at most one vertex in common,

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof. (a) ASSUME there are distinct blocks B_1 and B_2 of G with at least two common vertices. By Note 5.2.B, B_1 and B_2 are necessarily loopless. Because they are distinct maximal nonseparable subgraphs of G, neither one contains the other. Hence $B = B_1 \cup B_2$ properly contains both B_1 and B_2.

Proposition 5.3

Proposition 5.3. Let G be a graph. Then:

(a) any two blocks of G have at most one vertex in common,

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof. (a) ASSUME there are distinct blocks B_1 and B_2 of G with at least two common vertices. By Note 5.2.B, B_1 and B_2 are necessarily loopless. Because they are distinct maximal nonseparable subgraphs of G, neither one contains the other. Hence $B = B_1 \cup B_2$ properly contains both B_1 and B_2. Let $v \in V(B)$. Then $B - v = (B_1 - v) \cup (B_2 - v)$ is connected because $B_1 - v$ and $B_2 - v$ are both connected (since B_1 and B_2 are blocks, then they are by definition nonseparable and so v is not a separating vertex and hence is not a cut vertex) and $B_1 - v$ and $B_2 - v$ have at least one common vertex by our assumption.

Proposition 5.3

Proposition 5.3. Let G be a graph. Then:

(a) any two blocks of G have at most one vertex in common,

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof. (a) ASSUME there are distinct blocks B_1 and B_2 of G with at least two common vertices. By Note 5.2.B, B_1 and B_2 are necessarily loopless. Because they are distinct maximal nonseparable subgraphs of G, neither one contains the other. Hence $B = B_1 \cup B_2$ properly contains both B_1 and B_2. Let $v \in V(B)$. Then $B - v = (B_1 - v) \cup (B_2 - v)$ is connected because $B_1 - v$ and $B_2 - v$ are both connected (since B_1 and B_2 are blocks, then they are by definition nonseparable and so v is not a separating vertex and hence is not a cut vertex) and $B_1 - v$ and $B_2 - v$ have at least one common vertex by our assumption.

Proposition 5.3 (continued 1)

Proposition 5.3. Let G be a graph. Then:

(a) any two blocks of G have at most one vertex in common,

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof (continued). Since B_1 and B_2 are blocks, no vertex of B_i is a cut vertex of B_i for $i \in \{1, 2\}$ (as just described) and hence no vertex of B is a cut vertex of B. So B is a loopless connected graph with no cut vertices (and hence, by Note 5.2.A, no separating vertices) so that B is nonseparable. But this CONTRADICTS the fact that blocks B_1 and B_2 are maximal nonseparable graphs. So the assumption that B_1 and B_2 have two common vertices is false. So two blocks of G have at most one vertex in common, as claimed.

Proposition 5.3 (continued 1)

Proposition 5.3. Let G be a graph. Then:

(a) any two blocks of G have at most one vertex in common,

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof (continued). Since B_1 and B_2 are blocks, no vertex of B_i is a cut vertex of B_i for $i \in \{1, 2\}$ (as just described) and hence no vertex of B is a cut vertex of B. So B is a loopless connected graph with no cut vertices (and hence, by Note 5.2.A, no separating vertices) so that B is nonseparable. But this CONTRADICTS the fact that blocks B_1 and B_2 are maximal nonseparable graphs. So the assumption that B_1 and B_2 have two common vertices is false. So two blocks of G have at most one vertex in common, as claimed.

Proposition 5.3 (continued 2)

Proposition 5.3. Let G be a graph. Then:

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof (continued). **(b)** Each edge of G induces a nonseparable subgraph of G (either a loop on one vertex or a K_2 on two vertices). So each edge of G is contained in some maximal nonseparable subgraph of G; that is, each edge of G is contained in some block of G. So the union of the (edge sets of the) blocks of G gives (the edge set of) G itself (so the blocks form a cover of G).

Proposition 5.3 (continued 2)

Proposition 5.3. Let G be a graph. Then:

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof (continued). (b) Each edge of G induces a nonseparable subgraph of G (either a loop on one vertex or a K_2 on two vertices). So each edge of G is contained in some maximal nonseparable subgraph of G; that is, each edge of G is contained in some block of G. So the union of the (edge sets of the) blocks of G gives (the edge set of) G itself (so the blocks form a cover of G). By part (a), no two blocks can have two vertices in common so no two blocks can share a (nonloop) edge. If G has loops, then each single loop is a block and so *distinct* blocks cannot share a loop. So the blocks are edge (and loop) disjoint so that the blocks form a (edge) decomposition of G, as claimed.

Proposition 5.3 (continued 2)

Proposition 5.3. Let G be a graph. Then:

(b) the blocks of G form a (edge) decomposition of G,

(c) each cycle of G is contained in a block of G.

Proof (continued). (b) Each edge of G induces a nonseparable subgraph of G (either a loop on one vertex or a K_2 on two vertices). So each edge of G is contained in some maximal nonseparable subgraph of G; that is, each edge of G is contained in some block of G. So the union of the (edge sets of the) blocks of G gives (the edge set of) G itself (so the blocks form a cover of G). By part (a), no two blocks can have two vertices in common so no two blocks can share a (nonloop) edge. If G has loops, then each single loop is a block and so *distinct* blocks cannot share a loop. So the blocks are edge (and loop) disjoint so that the blocks form a (edge) decomposition of G, as claimed.

Proposition 5.3 (continued 3)

Proposition 5.3. Let G be a graph. Then:

(c) each cycle of G is contained in a block of G.

Proof (continued). **(c)** A cycle of G is a nonseparable subgraph of G (by Theorem 5.2, say), and so is contained in some maximal nonseparable subgraph of G. That is, each cycle is contained in a block of G, as claimed. □

Proposition 5.3 (continued 3)

Proposition 5.3. Let G be a graph. Then:

(c) each cycle of G is contained in a block of G.

Proof (continued). (c) A cycle of G is a nonseparable subgraph of G (by Theorem 5.2, say), and so is contained in some maximal nonseparable subgraph of G. That is, each cycle is contained in a block of G, as claimed. □

Theorem 5.4

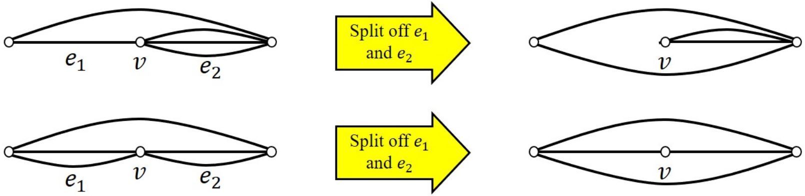
Theorem 5.4. THE SPLITTING LEMMA.
Let G be a nonseparable graph and let v be a vertex of G of degree at least four with at least two distinct neighbors. Then some two nonparallel edges incident to v can be split off so that the resulting graph is connected and has no cut edges.

Proof. Since G is nonseparable then it has no loops. There are two graphs on 3 vertices and 5 edges which satisfy the hypotheses of the lemma:

Theorem 5.4

Theorem 5.4. THE SPLITTING LEMMA.
Let G be a nonseparable graph and let v be a vertex of G of degree at least four with at least two distinct neighbors. Then some two nonparallel edges incident to v can be split off so that the resulting graph is connected and has no cut edges.

Proof. Since G is nonseparable then it has no loops. There are two graphs on 3 vertices and 5 edges which satisfy the hypotheses of the lemma:

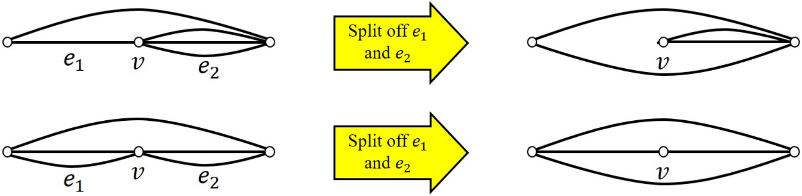


As can be seen here, these graphs satisfy the conclusion of the Splitting Lemma.

Theorem 5.4

Theorem 5.4. THE SPLITTING LEMMA.
Let G be a nonseparable graph and let v be a vertex of G of degree at least four with at least two distinct neighbors. Then some two nonparallel edges incident to v can be split off so that the resulting graph is connected and has no cut edges.

Proof. Since G is nonseparable then it has no loops. There are two graphs on 3 vertices and 5 edges which satisfy the hypotheses of the lemma:



As can be seen here, these graphs satisfy the conclusion of the Splitting Lemma.

Theorem 5.4 (continued 1)

Proof (continued). Let f be an edge of G not incident to vertex v and set $H = G \setminus f$. If v is an internal vertex of some block of H, then the result follows by induction on the number of edges of the block, as is to be shown in Exercise 5.2.A. So we consider the case where v is not an interval vertex of a block of H; that is, we let v be a separating vertex of H (and hence a cut vertex of H, by Note 5.2.A). Since G is nonseparable by hypothesis, by Exercise 5.2.11 we have that the block tree of $H = G \setminus f$ is a path. Since G is nonseparable (and hence has no cut vertices) then f must link internal vertices of the two end blocks of H (otherwise G would have cut vertices):

Theorem 5.4 (continued 1)

Proof (continued). Let f be an edge of G not incident to vertex v and set $H = G \setminus f$. If v is an internal vertex of some block of H, then the result follows by induction on the number of edges of the block, as is to be shown in Exercise 5.2.A. So we consider the case where v is not an interval vertex of a block of H; that is, we let v be a separating vertex of H (and hence a cut vertex of H, by Note 5.2.A). Since G is nonseparable by hypothesis, by Exercise 5.2.11 we have that the block tree of $H = G \setminus f$ is a path. Since G is nonseparable (and hence has no cut vertices) then f must link internal vertices of the two end blocks of H (otherwise G would have cut vertices):

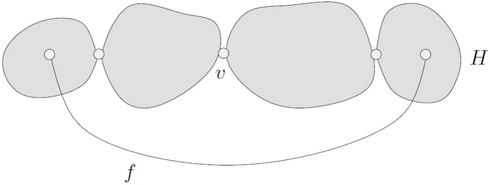


Figure 5.2(a) The block path H and the edge f

Theorem 5.4 (continued 1)

Proof (continued). Let f be an edge of G not incident to vertex v and set $H = G \setminus f$. If v is an internal vertex of some block of H, then the result follows by induction on the number of edges of the block, as is to be shown in Exercise 5.2.A. So we consider the case where v is not an interval vertex of a block of H; that is, we let v be a separating vertex of H (and hence a cut vertex of H, by Note 5.2.A). Since G is nonseparable by hypothesis, by Exercise 5.2.11 we have that the block tree of $H = G \setminus f$ is a path. Since G is nonseparable (and hence has no cut vertices) then f must link internal vertices of the two end blocks of H (otherwise G would have cut vertices):

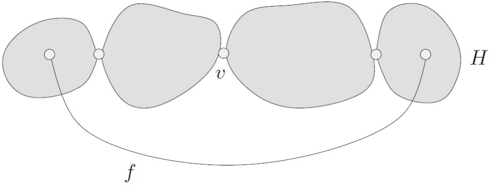


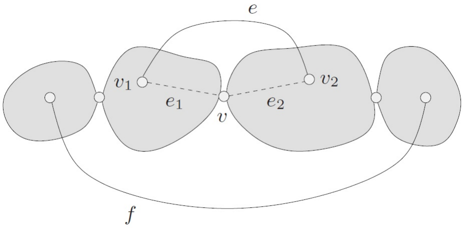
Figure 5.2(a) The block path H and the edge f

Theorem 5.4 (continued 2)

Proof (continued). Let e_1 and e_2 be two edges incident with v which lie in distinct blocks of H (such edges exist since the block tree of H is a path, so there is a block of H "to the left of v" and a block of H "to the right of v"). Consider the graph G' derived from G by spitting off e_1 and e_2, shown below in Figure 5.2(b). In Exercise 5.2.9 it is to be shown that G' is connected and that each edge of G' lies in a cycle. Then by Proposition 3.2, G' has no cut edges. Then edges e_1 and e_2 are the nonparallel edges of G claimed to exist and G' is the graph in which these edges have been split off, and G' is connected and has no cut edges, as claimed. □

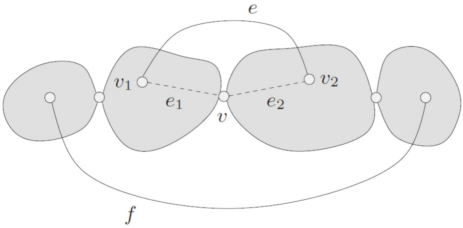
Theorem 5.4 (continued 2)

Proof (continued). Let e_1 and e_2 be two edges incident with v which lie in distinct blocks of H (such edges exist since the block tree of H is a path, so there is a block of H "to the left of v" and a block of H "to the right of v"). Consider the graph G' derived from G by spitting off e_1 and e_2, shown below in Figure 5.2(b). In Exercise 5.2.9 it is to be shown that G' is connected and that each edge of G' lies in a cycle. Then by Proposition 3.2, G' has no cut edges. Then edges e_1 and e_2 are the nonparallel edges of G claimed to exist and G' is the graph in which these edges have been split off, and G' is connected and has no cut edges, as claimed. □



Theorem 5.4 (continued 2)

Proof (continued). Let e_1 and e_2 be two edges incident with v which lie in distinct blocks of H (such edges exist since the block tree of H is a path, so there is a block of H "to the left of v" and a block of H "to the right of v"). Consider the graph G' derived from G by spitting off e_1 and e_2, shown below in Figure 5.2(b). In Exercise 5.2.9 it is to be shown that G' is connected and that each edge of G' lies in a cycle. Then by Proposition 3.2, G' has no cut edges. Then edges e_1 and e_2 are the nonparallel edges of G claimed to exist and G' is the graph in which these edges have been split off, and G' is connected and has no cut edges, as claimed. □



Theorem 5.5

Theorem 5.5. The Cycle Double Cover Conjecture is true if and only if it is true for all nonseparable cubic graphs.

Proof. As observed above, Proposition 5.3(b and c) a graph has a cycle double cover if and only if each of its blocks has a cycle double cover. So it suffices to consider the Cycle Double Cover Conjecture for blocks (i.e., nonseparable graphs). Let G be a nonseparable graph (so G has no loops by Note 5.2.B).

Theorem 5.5

Theorem 5.5. The Cycle Double Cover Conjecture is true if and only if it is true for all nonseparable cubic graphs.

Proof. As observed above, Proposition 5.3(b and c) a graph has a cycle double cover if and only if each of its blocks has a cycle double cover. So it suffices to consider the Cycle Double Cover Conjecture for blocks (i.e., nonseparable graphs). Let G be a nonseparable graph (so G has no loops by Note 5.2.B). By Veblen's Theorem (Theorem 2.7), if G is even then it admits a cycle decomposition and hence it has a cycle double cover (just take two copies of the cycle decomposition). So without loss of generality, we may assume that G has at least one vertex of odd degree. We now consider two "manipulations" of a nonseparable graph.

Theorem 5.5

Theorem 5.5. The Cycle Double Cover Conjecture is true if and only if it is true for all nonseparable cubic graphs.

Proof. As observed above, Proposition 5.3(b and c) a graph has a cycle double cover if and only if each of its blocks has a cycle double cover. So it suffices to consider the Cycle Double Cover Conjecture for blocks (i.e., nonseparable graphs). Let G be a nonseparable graph (so G has no loops by Note 5.2.B). By Veblen's Theorem (Theorem 2.7), if G is even then it admits a cycle decomposition and hence it has a cycle double cover (just take two copies of the cycle decomposition). So without loss of generality, we may assume that G has at least one vertex of odd degree. We now consider two "manipulations" of a nonseparable graph.

Theorem 5.5 (continued 1)

Proof (continued).
Manipulation 1. Suppose G has a vertex v of degree 2 with distinct neighbors u and w (if v is degree two and neighbors u and w are equal then $u = w$ is a cut vertex, so since G is nonseparable, this cannot be the case). Let G' be the graph obtained from $G - v$ by adding a new edge joining u and w (notice that this is not splitting off the edges uv and vw since vertex v is deleted here; this is the inverse operation of subdividing an edge). Since G is nonseparable then it contains no loops (by Note 5.2.B) and no cut vertices. Notice that G' also contains no loops (u and w are distinct) and no cut vertices (a vertex of G' would also be a cut vertex of G). That is, G' is nonseparable.

Theorem 5.5 (continued 1)

Proof (continued).
Manipulation 1. Suppose G has a vertex v of degree 2 with distinct neighbors u and w (if v is degree two and neighbors u and w are equal then $u = w$ is a cut vertex, so since G is nonseparable, this cannot be the case). Let G' be the graph obtained from $G - v$ by adding a new edge joining u and w (notice that this is not splitting off the edges uv and vw since vertex v is deleted here; this is the inverse operation of subdividing an edge). Since G is nonseparable then it contains no loops (by Note 5.2.B) and no cut vertices. Notice that G' also contains no loops (u and w are distinct) and no cut vertices (a vertex of G' would also be a cut vertex of G). That is, G' is nonseparable.

Manipulation 2. If G has a vertex v of degree 4 or more, let G' be the graph obtained from G by splitting off two edges incident to v. As in Manipulation 1, G' is nonseparable.

Theorem 5.5 (continued 1)

Proof (continued).
Manipulation 1. Suppose G has a vertex v of degree 2 with distinct neighbors u and w (if v is degree two and neighbors u and w are equal then $u = w$ is a cut vertex, so since G is nonseparable, this cannot be the case). Let G' be the graph obtained from $G - v$ by adding a new edge joining u and w (notice that this is not splitting off the edges uv and vw since vertex v is deleted here; this is the inverse operation of subdividing an edge). Since G is nonseparable then it contains no loops (by Note 5.2.B) and no cut vertices. Notice that G' also contains no loops (u and w are distinct) and no cut vertices (a vertex of G' would also be a cut vertex of G). That is, G' is nonseparable.

Manipulation 2. If G has a vertex v of degree 4 or more, let G' be the graph obtained from G by splitting off two edges incident to v. As in Manipulation 1, G' is nonseparable.

Theorem 5.5 (continued 2)

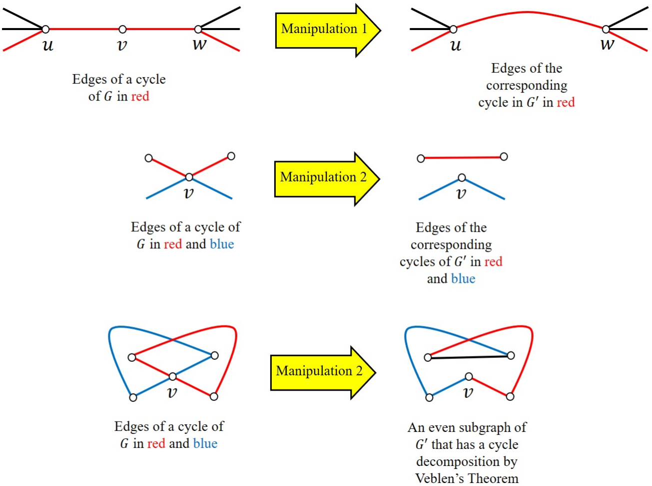
Proof (continued). In Manipulation 1, a vertex of degree 2 is eliminated and all other vertices remain the same degree. In Manipulation 2, a vertex of degree 4 or greater is modified in such a way that its degree is reduced by 2 and the degrees of all other vertices remain the same. In the application of either Manipulation, a nonseparable graph is produced. Notice that a nonseparable graph cannot have a vertex of degree 1 (a pendant edge would represent a bong of a graph and so a nonseparable graph cannot have a pendant edge). So by repeatedly applying Manipulations 1 and 2, a nonseparable cubic graph H is produced (since G has at least one vertex of odd degree, repeatedly applying the Manipulations does not result in the null graph). A cycle double cover of H can then be used to produce a cycle double cover of G and conversely, as claimed. This last claim is illustrated below.

Theorem 5.5 (continued 2)

Proof (continued). In Manipulation 1, a vertex of degree 2 is eliminated and all other vertices remain the same degree. In Manipulation 2, a vertex of degree 4 or greater is modified in such a way that its degree is reduced by 2 and the degrees of all other vertices remain the same. In the application of either Manipulation, a nonseparable graph is produced. Notice that a nonseparable graph cannot have a vertex of degree 1 (a pendant edge would represent a bong of a graph and so a nonseparable graph cannot have a pendant edge). So by repeatedly applying Manipulations 1 and 2, a nonseparable cubic graph H is produced (since G has at least one vertex of odd degree, repeatedly applying the Manipulations does not result in the null graph). A cycle double cover of H can then be used to produce a cycle double cover of G and conversely, as claimed. This last claim is illustrated below.

Theorem 5.5 (continued 3)

Proof (continued).



□