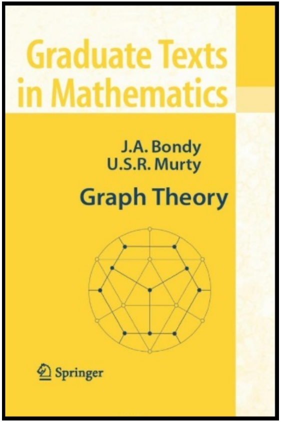
Graph Theory

Chapter 5. Nonseparable Graphs

5.3. Ear Decompositions—Proofs of Theorems



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Proposition 5.6

Proposition 5.6. Let F be a nontrivial proper subgraph of a nonseparable graph G. Then there is an ear of F in G.

Proof. First, suppose F is a spanning subgraph of G. Then $E(G) \setminus E(F)$ is nonempty because, by hypothesis, F is a proper subgraph of G. Every edge in $E(G) \setminus E(F)$ is then an ear of F in G, as claimed. So without loss of generality we now assume F is not a spanning subgraph of G.

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Since G is connected then (by the definition of connected) for sets $V(F)$ and $V(G) \setminus V(F)$, which partition $V(G)$, there is an edge xy of G with $x \in V(F)$ and $y \in V(G) \setminus V(F)$. Since G is nonseparable by hypothesis then it has no separating vertices and hence no cut vertices by Note 5.2.A. So $G - x$ is connected and by Exercise 3.1.4 there is a path Q in $G - x$ with one end in $\{y\}$, one end in $V(F - x)$, and all interior vertices in $F - \{x, y\}$. Then path xyQ has $x \in V(F)$ as an end, and end in $V(F - x) \subset V(F)$, and no interior vertices of $P = xyQ$ are in F. That is, P is an ear of F in G, and so the claim holds. □

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Theorem 5.8

Theorem 5.8. Every nonseparable graph other then K_1 and K_2 has an ear decomposition.

Proof. Since neither K_1 nor K_2 contains a cycle, then neither has an ear decomposition. Let G be a nonseparable graph other than K_1 or K_2. Since a nonseparable graph G is connected by definition, then by Theorem 5.2 G contains a cycle G_0. Let $F_1 = G_0$. If $F_1 \neq G$, then F_1 is a proper nontrivial subgraph of nonseparable graph G, and so there is an ear P_1 of F_1 in G by Proposition 5.6. Since F_1 is a cycle then by Theorem 5.2 it is nonseparable. Define $G_1 = F_1 \cup P_1$. Then G_1 is nonseparable by Proposition 5.7.

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Theorem 5.10

Theorem 5.10. Every connected graph without cut edges has a strong orientation.

Proof. Let G be a connected graph without cut edges. By Proposition 5.9, it suffices to show that each block B of G has a srong orientation. We may assume $B \neq K_1$ since K_1 vacuously has a strong orientation. Because G has no cut edges then we have $B \neq K_2$. So by Theorem 5.2 B contains a cycle, and by Theorem 5.8 B has an ear decomposition $(G_0, G_1, \ldots, G_k)$. Consider the orientation of B obtained by orienting G_0 as a directed cycle and orienting each ear as a directed path. Since each vertex of a directed cycle is of out-degree 1 and each internal vertex of a directed path is of out-degree 1, then $\partial^+(X) \neq \varnothing$ for X any subset of the orientation of $V(G_1)$. Therefore the orientations of G_0 and G_1 are strongly connected.

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Theorem 5.10 (continued)

Theorem 5.10. Every connected graph without cut edges has a strong orientation.

Proof (continued). If the orientation of G_i is strongly connected then $\partial^+(X) \neq \varnothing$ for all subsets of $V(G_i)$ and in particular the out-degrees of each vertex of the orientation of G_i are positive. Since $G_{i+1} = G_i \cup P_i$ where P_i is an ear of G_i in G, then each vertex of the orientation of G_i is of positive degree and each internal vertex of the orientation of P_i is of out-degree 1. Therefore $\partial^+(X) \neq \varnothing$ for all subsets of $V(G_{i+1})$ and hence the orientation of G_{i+1} is strongly connected and by induction the orientations of G_i are all strongly connected for $0 \leq i \leq k$. That is, block G_k has a strong orientation and the claim now follows from Proposition 5.9. □