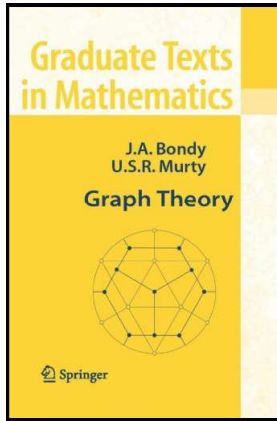


Graph Theory

Chapter 6. Tree-Search Algorithms 6.1. Tree-Search—Proofs of Theorems



Theorem 6.2

Theorem 6.2. Let T be a BFS-tree of a connected graph G , with root r . Then:

- (a) for every vertex v of G , $\ell(v) = d_T(r, v)$, the level of v in T , and
- (b) every edge of G joins vertices on the same or consecutive levels of T ; that is, $|\ell(u) - \ell(v)| \leq 1$ for all $uv \in E$.

Proof. (a) This is to be proved (inductively) in Exercise 6.1.1.

(b) If uv is an edge of G joining u and v on the same level, then by the definition of "level," $\ell(u) = \ell(v)$ as claimed. So it suffices to prove that if $uv \in E$ and $\ell(u) < \ell(v)$ then $\ell(u) = \ell(v) - 1$.

First, we prove by induction on $\ell(u)$ that if u and v satisfy $\ell(u) < \ell(v)$, then u is joined to A "before" (in terms of the values of $t(u)$ and $t(v)$). For the base case, if $\ell(u) = 0$ then $u = r$ is the root of the tree, $t(u) = 1$, and for every other vertex $i > 1$ (by Step 9) and so every other vertex v satisfies $t(v) > 1$ (by Step 11).

Theorem 6.2 (continued 1)

Proof (continued). For the induction hypothesis, suppose that the assertion is true whenever $\ell(u) < k$, and consider the case $\ell(u) = k$ where $k > 0$. Notice that $0 < k = \ell(u) < \ell(v)$ so that neither u nor v is the root. Use the predecessor function p to define $x = p(u)$ and $y = p(v)$. Since predecessors in a rooted tree are unique, then $x \neq y$. By Step 11 we have $\ell(x) = \ell(u) - 1$ (with y of Step 11 equal to u here) and $\ell(y) = \ell(v) - 1$ (with x and y of Step 11 equal to y and v , respectively, here). Since we consider $\ell(u) < \ell(v)$, then we have $\ell(x) = \ell(u) - 1 < \ell(v) - 1 = \ell(y)$. Since $\ell(u) = k$ then $\ell(x) = k - 1 < k$ and so by the induction hypothesis we have that x joined Q before y (i.e., $t(x) < t(y)$). Since u is a neighbor of x in T (by the predecessor function) and $u \neq v$ then u joined Q before v (i.e., $t(u) < t(v)$). Therefore, by mathematical induction, if $\ell(u) < \ell(v)$ then u joined Q before v .

Theorem 6.2 (continued 2)

Proof (continued). Now consider $uv \in E$ with $\ell(u) < \ell(v)$. (i) If $u = p(v)$, then by Step 11 we have $\ell(u) = \ell(v) - 1$ (with x and y of Step 11 equal to u and v , respectively, here), and the claim follows. (ii) If $u \neq p(v)$ then set $y = p(v)$. Then v was added to T by the edge yv (by Steps 11 and 12) and not by the edge uv . So vertex y joined Q before u , for if u had joined Q before y (and hence before v), then the fact that v is a neighbor of u means that v would have joined Q before y by Step 8 (with v as an uncoloured neighbor of u) and Step 12 (with y there equal to v here), contradicting the fact that y is a predecessor of v . So by the first part of the proof of (b), $\ell(y) \leq \ell(u)$. Since $y = p(v)$ then $\ell(v) - 1 = \ell(y)$ (by Step 11) and since we are considering edge $uv \in E$ with $\ell(u) < \ell(v)$ (and hence $\ell(u) \leq \ell(v) - 1$ since ℓ is integer valued), then we have $\ell(v) - 1 = \ell(y) \leq \ell(u) \leq \ell(v) - 1$. This gives $\ell(u) = \ell(v) - 1$, as claimed. So for uv an edge of G , we have either $\ell(u) = \ell(v)$ (when u and v are at the same level) or $\ell(u) = \ell(v) - 1$ in the case that $\ell(u) < \ell(v)$. That is, $|\ell(u) - \ell(v)| \leq 1$ for all $uv \in E$. □

Theorem 6.3

Theorem 6.3. Let G be a connected graph. Then the values of the level function ℓ returned by BFS are the distances in G from the root r : $\ell(v) = d_G(r, v)$ for all $v \in V$.

Proof. By Theorem 6.2(a), $\ell(v) = d_T(r, v)$. Now $d_T(r, v) \geq d_G(r, v)$ because T is a subgraph of G . Thus $\ell(v) \geq d_G(r, v)$. We now reverse this inequality by an induction argument on the length of the shortest (r, v) -path in G .

Let P be a shortest (r, v) -path in G , where $r \neq v$. If P is of length 1 then v is adjacent to r in G . By Steps 3 through 12, the root r is first added to Q and then all neighbors of r are added to T at level 1. This establishes the base case for the induction argument. Now, for the induction hypothesis, suppose the inequality $\ell(v) \leq d_G(r, v)$ holds for shortest (r, v) -path P in G of length k . Consider vertex v in G such that the shortest (r, v) -path P in G is of length $k + 1$. Let u be the predecessor of v on P (not to be confused with the predecessor $p(v)$ in T).

Theorem 6.3 (continued)

Theorem 6.3. Let G be a connected graph. Then the values of the level function ℓ returned by BFS are the distances in G from the root r : $\ell(v) = d_G(r, v)$ for all $v \in V$.

Proof (continued). Then rPu is a shortest (r, u) -path in G , and $d_G(r, u) = d_G(r, v) = 1 = k$. So by the induction hypothesis, $\ell(u) \leq d_G(r, u)$. Since u and v are adjacent in G , then by Theorem 6.2(b) we have $\ell(v) - \ell(u) \leq 1$. Therefore $\ell(v) \leq \ell(u) + 1 \leq d_G(r, u) + 1 = d_G(r, v)$.

So the induction step holds and by mathematical induction $\ell(v) \leq d_G(r, v)$ for all vertices v in G . Combining this with the first inequality gives the desired equality. \square

Proposition 6.5

Proposition 6.5. Let u and v be two vertices of G , with $f(u) < f(v)$.

- (a) If u and v are adjacent in G , then $l(u) < l(v)$.
- (b) u is an ancestor of v in T if and only if $l(v) < l(u)$.

Proof. (a) Informally, this claim is that if u is added to the top of stack S before its neighbor v (i.e., $f(u) < f(v)$), then v leaves the stack before u (i.e., $l(u) < l(v)$). By Steps 12 and 13, u is added to stack S at time $f(u)$. By Steps 10–13, all uncolored neighbors are considered for addition to S before vertex u is removed from S . Parameter i is incremented by 1 in Step 9 after each uncolored neighbor of u is considered for addition to S . Since $f(u) < f(v)$ then v is one of these neighbors of u . So v is added to the top of stack S while u is still in S . Since vertices are removed from the top of S (by Steps 8 and 16), and i is incremented by 1 each time a vertex is removed from the stack (by Step 9), so the l value of v (assigned in Step 15) is less than the l value of u , $l(v) < l(u)$, as claimed.

Proposition 6.5 (continued 1)

Proposition 6.5. Let u and v be two vertices of G , with $f(u) < f(v)$.

- (a) If u and v are adjacent in G , then $l(u) < l(v)$.
- (b) u is an ancestor of v in T if and only if $l(v) < l(u)$.

Proof (continued). (b) First, suppose that u is an ancestor of v in T . By Steps 9 and 12, the values of f increase along the path uTv (i is incremented in Step 9 and f values are assigned to vertices using the value of i and the predecessor function in Step 12). That is, $f(u) < f(v)$. So by part (a) to each vertex in path uTv we have $l(u) < l(v)$ (we have to apply to consecutive vertices in the path since part (a) requires that we compare neighbors), as claimed.

Conversely, suppose that u is not an ancestor of v in T . Since $f(u) < f(v)$ by hypothesis then v is not an ancestor of u either, it could lie on another “branch” of the family tree). So u does not lie on the path rTv and v does not lie on the path rTu . Let s be the last common vertex of these two paths.

Proposition 6.5 (continued 2)

Proposition 6.5. Let u and v be two vertices of G , with $f(u) < f(v)$.

- (a) If u and v are adjacent in G , then $l(u) < l(v)$.
- (b) u is an ancestor of v in T if and only if $l(v) < l(u)$.

Proof (continued). Since $f(u) < f(v)$ then u was added to stack S before v , and hence all the proper descendants of s on path rTv can be added to S only after all the proper descendants of s on the path rTu have been added and removed from S (after which s is the top vertex). In particular, v can only be added to S (at time $f(v)$) after u has been removed (at time $l(u)$), so that $l(u) < f(v)$. Since v is added to S before it is removed then $f(v) < l(v)$. Therefore $l(u) < f(v) < l(v)$. So if u is not an ancestor of v in T then $l(u) < l(v)$. Since $u \neq v$ then $l(u) \neq l(v)$, so the contrapositive of what we have shown is that if $l(u) \geq l(v)$ (that is, if $l(v) < l(u)$) then u is an ancestor of v in T , as claimed. \square

Proposition 6.6

Proposition 6.6. Let T be a DFS-tree of a graph G . Then every edge of G joins vertices which are related in T .

Proof. Let uv be any edge of G . Without loss of generality, say $f(u) < f(v)$. By Proposition 6.5(a) we have $l(v) < l(u)$. By Proposition 6.5(b), u is an ancestor of v and so (by definition) u and v are related in T . \square